Mechanics of Materials.

CHURCH.

MECHANICS
OF THE
STRENGTH AND ELASTICITY
OF
SOLIDS

Being Part III^d of the MECHANICS OF ENGINEERING *by Irving P. Church, C. E.*

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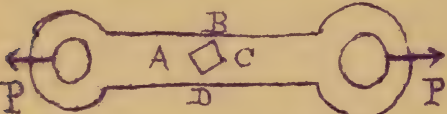

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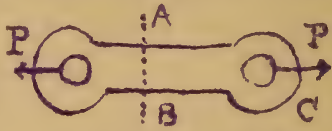

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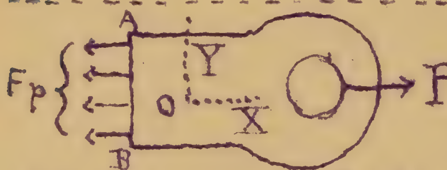

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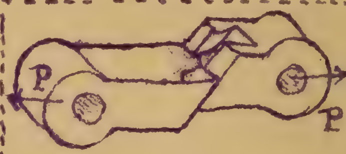

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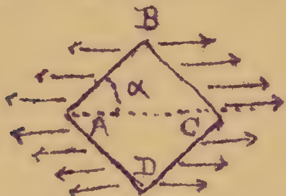

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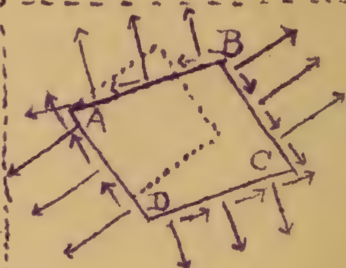

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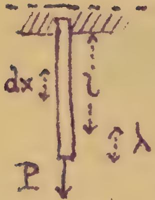

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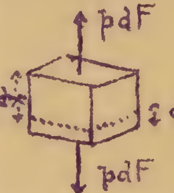

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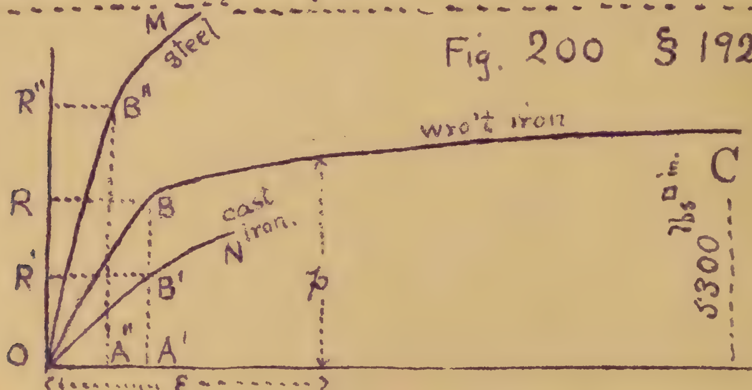


Fig. 200 § 192.

PART III.
STRENGTH OF MATERIALS.

Chap. I. Elementary Stresses & Strains.

178. DEFORMATION OF SOLID BODIES. In the preceding portions of this work, what was called technically a "rigid body" was supposed incapable of changing its form, i.e. the positions of its particles relatively to each other, under the action of any forces to be brought upon it. This supposition was made because the change of form which must actually occur does not appreciably alter the distances, angles, angles, etc., measured in any one body, among most of the pieces of a properly designed structure or machine. To show how the individual pieces of such constructions should be designed to avoid undesirable deformation or injury is the object of this division of Mechanics of Engineering, viz., the Strength of Materials.

As perhaps the simplest instance of the deformation or distortion of a solid, let us consider the case of a prismatic rod in a state of tension, Fig. 192 (link of a surveyor's chain, e.g.). The pull at each end is P, and the body is said to be under a tension of P (lbs., tons, or other unit), not 2P. Let ABCD be the end view of an elementary parallelopiped, originally of square section and with faces at 45° with the axis of the prism. It is now deformed, the four faces perpendicular to the paper being longer than before, while the angles BAD and BCD, originally right angles, are now smaller by a certain amount δ, ABC and ADC larger by an equal amount δ. The element is said to be in a state of STRAIN, viz.: the elongation of its edges (perpendicular to paper) is called a TENSILE STRAIN, while the alteration in the an-

gles between its faces is called a shearing strain, or angular distortion (sometimes also called sliding, or tangential, since BC has been made to slide, relatively to AD, and thereby caused the change of angle). [This use of the word strain, to signify change of form and not the force producing it, is of recent adoption among many, though not all, technical writers.]

179. STRAINS. TWO KINDS ONLY. Just as a curved line may be considered to be made up of small straight line elements, so the substance of any solid body may be considered to be made up of small contiguous parallelopipeds, whose angles are each 90° before the body is subjected to the action of forces but which are not necessarily cubes. A line of such elements forming an elementary prism is sometimes called a fibre, but this does not necessarily imply a fibrous nature in the material in question. The system of imaginary cutting surfaces by which the body is thus subdivided need not consist entirely of planes; in the subject of Torsion, for instance, the parallelopipedical elements considered lie in concentric cylindrical shells, cut both by transverse and radial planes.

Since these elements are taken so small that the only possible changes of form in any one of them, as induced by a system of external forces acting on the body, are elongations or contractions of its edges, and alteration of its angles, there are but two kinds of strain ELONGATION (contraction, if negative) and SHEARING.

180. DISTRIBUTED FORCES, or STRESSES. In the matter preceding this chapter it has sufficed for practical purposes to consider a force as applied at a point of a body, but in reality it must be distributed over a definite area; for otherwise the material would be subjected to an infinite force per unit of area. (Forces like gravity, magnetic attraction, etc. we have already treated as distributed over the mass of a body, but reference is now had particularly to the pressure of one body against another, or

the action of one portion of the body on the remainder.) For instance, sufficient surface must be provided between the end of a loaded beam and the pier on which it rests to avoid injury to either. Again too small a wire must not be used to sustain a given load, or the tension per unit of area of its cross section becomes sufficient to rupture it.

<u>Stress</u> <u>is</u> <u>distributed force</u>, and its intensity at any point of the area is

$$p = \frac{dP}{dF} \qquad (1)$$

where dF is a small area containing the point and dP the force coming upon that area. If equal dP's (all parallel) act on equal dF's of a plane surface, the stress is said to be of uniform intensity, which is then

$$p = \frac{P}{F} \qquad (2)$$

where P = total force and F the total area over which it acts. The steam pressure on a piston is an example of stress of uniform intensity.

181. STRESSES ON AN ELEMENT; OF TWO KINDS ONLY. When a solid body of any material is in equilibrium under a system of forces which do not rupture it, not only is its shape altered (i.e. its elements are <u>strained</u>), and stresses produced on those planes on which the forces act, but other stresses also are induced on some or all internal surfaces which separate element from element, (over and above the forces with which the elements may have acted on each other before the application of the external stresses or "applied forces"). So long as the whole solid is the "<u>free body</u>" under consideration, these "internal stresses, being the forces with which the portion on one side of any imaginary cutting plane acts on the portion on the other side, do not appear in any equation of equilibrium (for if introduced they would cancel out); but if we consider free a portion only, some or all of whose bounding surfaces are cutting planes of the original body, the stresses existing on those planes are brought into the equations of equilibrium.

Similarly if a single element of the body is treated by itself, the stresses on all six of its faces, together with its weight, form a balanced system of forces, the body being supposed at rest.

As an example of internal stress, consider again the case of a rod in tension. Fig. 193 shows the whole rod (or eye-bar) free, the forces P being the pressures of the pins in the eyes, and causing external stress (compression here) on the surfaces of contact. Conceive a right section made through AB, far enough from the eye that we may consider the internal stress to be uniform in this section, and consider the portion ABC as a free body, in fig. 194. The stresses on AB, now one of the bounding surfaces of the free body, must be parallel to P, i.e., normal to AB; (otherwise they would have components perpendicular to P, which is precluded by the necessity of ΣY being $= 0$, and the supposition of uniformity.) Let F = the sectional area AB, and p = the stress per unit of area; then

$$\Sigma X = 0 \quad \text{gives} \quad P = Fp \;, \text{ i.e., } \; p = \frac{P}{F} \qquad (2)$$

The state of internal stress, then, is such that on planes perpendicular to the axis of the bar the stress is tensile and normal (to those planes). Since if a section were made oblique to the axis of the bar, the stress would still be parallel to the axis for reasons as above, it is evident that on an oblique section, the stress has components both normal and tangential to the section, the normal component being a tension.

The presence of the tangential or shearing stress in oblique sections is rendered evident by considering that if an oblique dove-tail joint were cut in the rod, Fig. 195, the shearing stress on its surfaces may be sufficient to overcome friction and cause sliding along the oblique plane.

If a short prismatic block is under the compressive action of two forces, each = P and applied centrally in one base, we may show that the state of internal stress is the same as that of the rod under tension except that the normal stresses are of contra-

ry sign, i.e. compressive instead of tensile, and that the shearing stresses (or tendency to slide) on oblique plains are opposite in direction to those in the rod.

Since the resultant stress on a given internal plane of a body is fully represented by its normal and tangential components we are therefore justified in considering but two kinds of internal stress, <u>normal</u> or <u>direct</u>, and <u>tangential</u> or <u>shearing</u>.

182. RELATION BETWEEN STRESS AND STRAIN. This is best apprehended in a particular example, that of a rod in Tension. Consider free a small cubic element whose edge $= a$ in length and which has two faces parallel to the paper, being taken near the middle of the rod in fig. 193. Let the angle which the face AB, fig. 196, makes with the axis of the rod be $= \alpha$. This angle, for our present purpose, is considered to remain unchanged while the two forces P are acting, as before their action. The resultant stress on the face AB being of an intensity $p = P \div F$, (see eq. 2) per unit of <u>transverse section</u> of rod, is $= p\,(a \sin\alpha)\,a$. Hence its component normal to AB is $pa^2\sin^2\alpha$ and the tangential component $= pa^2 \sin\alpha\cos\alpha$. Dividing by the area, a^2, we have the following:

For a rod in simple tension we have, on a plane making an angle α with the axis, :

a NORMAL STRESS $= p\sin^2\alpha$ per unit of area; and
a SHEARING " " $= p\sin\alpha\cos\alpha$ " " "

"Unit of area" here refers to the oblique plane in question, while p denotes the normal stress per unit of area of a transverse section, i.e., when $\alpha = 90°$, fig. 194.

The stresses on CD are the same in value as on AB while for BC and AD we substitute $90° - \alpha$ for α. Fig. 197a shows these normal and shearing stresses, and also ~~much~~ exaggerated, the strains or change of form of the element (see fig. 192) Now experiment shows that so long as the stresses are of such moderate value that the piece recovers its original form completely when the external

-orces which induce the stresses are removed, corresponding stresses and strains are proportional (Hooke's), or may be so considered for practical purposes. Before strain the edge AB was equal to a; as the forces P (fig. 193) are gradually increased the elongation or additional length of AB increases in the same ratio, as the normal stress on BC and AD per unit of area. A similar statement may be made concerning the elongation of AD and the stress corresponding to it i.e. the normal stress on DC and AB. As for the shearing stresses, since $p \sin\alpha \cos\alpha = p \sin(90° - \alpha) \cos(90° - \alpha)$ they are of the same intensity per unit of area on all four faces (this is true in any state of stress). They are evidently accountable for the shearing strain, i.e. for the angular distortion, or difference, δ, between 90° and the present value of each of the four angles. According to Hooke's law then, as P increases within the limit mentioned above, δ varies proportionally to $p \sin\alpha \cos\alpha$.

182 a. EXAMPLE. Supposing the rod in question were of a kind of wood in which a shearing stress of 200 lbs. per sq. inch along the grain, or a normal stress of 400 lbs. per sq. inch, perpendicular to a fibre-plane will produce rupture, required the value of α the angle which the grain must make with the axis that, as P increases, the danger of rupture from each source may be the same. This requires that $200 : 400 :: p \sin\alpha \cos\alpha : p \sin^2\alpha$ i.e. $\tan.\alpha$ must $= 2.000 \therefore \alpha = 63\frac{1}{2}°$. If the cross section of the rod is 2 sq. inches, the force P at each end necessary to produce rupture of either kind, when $\alpha = 63\frac{1}{2}°$ is found by putting $p \sin\alpha \cos\alpha = 200 \therefore p = 500.0$ lbs. per sq. inch. Whence, since $p = P \div F$, $P = 1000$ lbs. (Units inch and pound.)

183. ELASTICITY is the name given to the property which most materials have, to a certain extent, of regaining their original form when the external forces exceed a certain limit called the ELASTIC LIMIT, the recovery of original form

on the part of the elements is only partial, the permanent deformation being called the SET.

Although theoretically the elastic limit is a perfectly definite stage of stress, experimentally it is somewhat indefinite and is generally considered to be reached when the permanent set becomes well marked as the stresses are increased and the test piece is given ample time for recovery in the intervals of rest

The SAFE LIMIT of stress taken well within the elastic limit, determines the working strength or safe load of the piece under consideration. E.g., the tables of safe loads of the rolled wrought iron beams, for floors, of the New Jersey Steel and Iron Co, at Trenton, are computed on the theory that the greatest normal stress (tension or compression) occurring on any internal plane shall not exceed 12000 lbs. per sq. inch, nor the greatest shearing stress 4000 lbs. per sq. inch.

The ULTIMATE LIMIT is reached when rupture occurs.

184. The MODULUS OF ELASTICITY (sometimes called co-efficient of elasticity) is the number obtained by the stress per unit of area by the corresponding relative strain.

Thus a rod of wrought iron of $\frac{1}{2}$ sq. inch sectional area being subjected to a tension of $2\frac{1}{2}$ tons = 5000 lbs., it is found that a length which was six feet before tension is now = 6.002 ft. The relative longitudinal strain or elongation is then $\varepsilon = (0.002) \div 6 = 1 : 3000$ and the corresponding stress (being the normal stress on a transverse plane) has an intensity of

$$p_t = P \div F = 5000 \div \tfrac{1}{2} = 10\,000 \text{ lbs. per sq. inch}$$

Hence by definition the modulus of elasticity is (for tension)

$$E_t = p_t \div \varepsilon = 10\,000 \div \frac{1}{3000} = 30\,000\,000 \text{ lbs. per}$$

sq. inch. It will be noticed that since ε is an abstract number E_t is of the same quality as p_t, i.e., lbs. per sq. inch, or one dimension of force divided by two dimensions

of space. (In the subject of strength of materials the inch is the most convenient English linear unit, when the pound is the unit of force; sometimes the foot and ton are used together.)

The foregoing would be called the modulus of elasticity of wro wrought iron in tension, in the direction of the fibre as given by the experiment quoted. But by Hooke's law p and δ vary together, for a given direction in a given material, hence within the elastic limit E is constant for a given direction in a given material. Experiment conforms this approximately.

Similarly, the modulus of elasticity for compression E_c, in a given direction in a given material may be determined by experiments on short blocks, or on rods confined laterally to prevent flexure.

As to the modulus of elasticity for shearing, S_s, we divide the shearing stress per unit of area in the given direction by δ (in π-measure) the corresponding angular strain or distortion.

Unless otherwise specified, by modulus of elasticity will be meant a value derived from experiments conducted within the elastic limit, and thus, whether for normal stress or for shearing, is approximately constant for a given direction in a given substance.

185. ISOTROPES. This name is given to materials which are homogeneous as regards their elastic properties. In such a material the moduli of elasticity are individually the same for any direction. E.g., a rod of rubber cut out of a large mass will exhibit the same elastic behavior when subjected to tension, whatever its original position in the mass. Fibrous materials like wood and wrought iron are not isotropic; the direction of grain in the former must al-

ways be considered. The "piling" and welding of numerous small pieces of iron prevents the resultant forging from being isotropic.

186. RESILIENCE refers to the potential energy stored in a body held under external forces in a state of stress which does not pass the elastic limit. On its release from constraint by virtue of its elasticity it can perform a certain amount of work called the resilience, depending in amount upon the circumstances of each case and the nature of the material.

187. GENERAL PROPERTIES OF MATERIALS. In view of some definitions already made we may say that a material is _ductile_ when the ultimate limit is far removed from the elastic limit; that it is _brittle_, like glass and cast iron, when these limits are near together. A small modulus of elasticity means that a comparatively small force is necessary to produce a given change of form, and vice versa, but implies little or nothing concerning the stress or strain at the elastic limit; thus Weisbach gives E_c for wrought iron 28,000,000 double the E_c for cast iron, while the compressive stresses at the elastic limit are the same for both materials.

188. GENERAL PROBLEM OF ~~[illegible]~~ INTERNAL STRESS. This as treated in the mathematical Theory of Elasticity, developed by Lamé, Clapeyron and Poisson, may be stated as follows:

Given the original form of a body when free from stress, and certain coefficients depending on its elastic properties; _required the new position, the altered shape, and the intensity of the stress on each of the six faces, of every element of the body, when a given balanced system of forces is applied to the body._

Solutions, by this theory, of certain problems of the nature just given involve elaborate, intricate, and bulky analysis; but for practical purposes Navier's theories (1838) and others of more recent date are sufficiently exact, when their moduli are properly determined by experiments covering a wide range of cases and materials. These will be given in the present work, and are comparatively simple. In some cases graphic will be preferred to analytic methods as more simple and direct, and indeed for some problems are the only meth-

ods yet discovered for obtaining solutions. Again, experiment is relied on almost exclusively in dealing with bodies of certain forms under peculiar systems of forces, empirical formulae being based on the experiments made; e.g. the collapsing of boiler tubes, and in some degree the flexure of long columns.

189. CLASSIFICATION OF CASES. Although in any case whatever of the deformation of a solid body by a balanced system of forces acting on it normal (tensile or compressive) and shearing stresses are both developed in every element which is affected at all (according to the plane section considered) cases where the body is prismatic, the external system consisting of two equal and opposite forces one at each end of the piece and directed away from each other, are commonly called cases of TENSION; if the piece is a short prism with the same two terminal forces directed <u>toward</u> each other, the case is said to be one of COMPRESSION; a case similar to the last, but where the prism is quite long ("long column"), is a case of FLEXURE or bending as are also most cases where the "applied forces" are not directed along the axis of the piece. Riveted joints and "pin-connections" present cases of SHEARING; a twisted shaft one of TORSION. When the gravity forces due to the weights of the elements are also considered a combination of two or more of the foregoing general cases may occur.

In each case, as treated, the principal objects aimed at are, so to design the piece or its loading that the greatest stress, in whatever element it may occur, shall not exceed a safe value; and sometimes furthermore to prevent too great deformation on the part of the piece. The first object is to provide sufficient STRENGTH; the second sufficient STIFFNESS.

190. TEMPERATURE STRESSES. If a piece is under such constraint that it is not free to change its form with changes of temperature, external forces are induced, the stresses produced by which are called <u>temperature stresses</u>.

TENSION.

191. HOOKE'S LAW BY EXPERIMENT. As a typical experiment in the tension of a long rod of ductile metal such as wrought iron and the mild steels, the following table is quoted from Prof. Cotterill's "Applied Mechanics". The experiment is old, made by Hodgkinson for an English Railway Commission, but well adapted to the purpose. From the great length of the rod, which was of wrought iron and 0.517 in. in diameter, the portion whose elongation was observed being 49 ft. 2 in. long, the small increase in length below the elastic limit was readily measured. The successive loads were of such a value that the tensile stress $p = P \div F$, or normal stress per sq. in. in the transverse section, was made to increase by equal increments of 2667.5 lbs. per sq. in., its initial value. After each application of load the elongation was measured; and after the removal of the load, the permanent set, if any.

Table of elongations of a wrought iron rod, of a length = 49 ft. 2 in.				
p	λ	$\Delta\lambda$	$\varepsilon = \lambda \div l$	λ_1
Load (lbs. per square inch)	Elongation (inches)	Increment of Elongation	ε, the relative elongation (abstract number)	Permanent Set (inches)
1 × 2667.5	.0485	.0485	0.000082	
2 × "	.1095	.061	.000186	
3 × "	.1675	.058	.000283	0.0015
4 × "	.224	.0565	.000379	.002
5 × "	.2805	.0565	.000475	.0027
6 × "	.337	.0565	.000570	.003
7 × "	.393	.056	·	.004
8 × "	.452	.059	.000766	.0075
9 × "	.5155	.0635		.0195
10 × "	.598	.0825		.049
11 × "	.760	.162		.1545
12 × "	1.310	.550		.667
etc.				

Referring now to fig. 198, the notation is evident. P is the total load in any experiment, F the cross section of the rod; hence the normal stress on the transverse section is $p = P \div F$. When the loads are increased by equal increments, the corresponding increments of the elongation λ should also be equal if Hooke's law is true. It will be noticed in the table that this is very nearly true up to the 8th loading, i.e. that $\Delta\lambda$, the difference between two consecutive values of λ, is nearly constant. In other words the proposition holds good:

$$P : P_1 :: \lambda : \lambda_1$$

if P and P_1 are any two loads below the 8th, and λ and λ_1 the corresponding elongations.

The permanent set is just perceptible at the 3rd load, and increases rapidly after the 8th, as also the increment of elongation. Hence at the 8th load, which produces a tensile stress on the cross section of $p = 8 \times 2667.5 = 21340.0$ lbs. per sq. inch, the elastic limit is reached.

As to the state of stress of the individual elements, if we conceive such a sub-division of the rod that four edges of each element are parallel to the axis of the rod, we find that it is in equilibrium between two normal stresses on its end faces (fig. 199) of a value $= p\,dF = (P \div F)\,dF$ where dF is the horizontal section of the element. If dx was the original length, and $d\lambda$ the elongation produced by $p\,dF$, we shall have, since all the dx's of the length are equally elongated at the same time,

$$\frac{d\lambda}{dx} = \frac{\lambda}{l}$$

But $d\lambda \div dx$ is the relative elongation ε, and by definition (§184) the Modulus of Elasticity for tension, $E_t = p \div \varepsilon$

$$\therefore E_t = \frac{p}{\frac{d\lambda}{dx}} \; ; \text{ or } E_t = \frac{Pl}{F\lambda} \quad \text{- - - - - - -} \quad (1)$$

Eq. (1) enables us to solve problems involving the elongation of a prism under tension, so long as the elastic limit is not surpassed.

The values of E_t computed from experiments like those

just cited should be the same for any load under the elastic limit, if Hooke's law were accurately obeyed, but in reality they differ somewhat, especially if the material lacks homogeneity. In the present instance (see Table) we have from the

2^{nd} Exper. $E_t = p \div \varepsilon = 28\,680\,000$ lbs. per sq. in.
5^{th} " $E_t =$ " $= 28\,000\,000$ " "
8^{th} " $E_t =$ " $= 27\,848\,000$ " "

If similar computations were made beyond the elastic limit, i.e., beyond the 8^{th} Exper., the result would be much smaller, showing the material to be yielding much more readily.

192. STRAIN DIAGRAMS. If we plot the stresses per sq. inch (p) as ordinates of a curve, and the corresponding relative elongations (ε) as abscissas, we obtain a useful graphic representation of the results of experiment.

Thus, the table of experiments just cited being utilized in this way, we obtain on paper a series of points through which a smooth curve may be drawn, viz; OBC Fig. 200, for wrought iron. Any convenient scales may be used for p and ε; and experiments having been made on other metals in tension and the results plotted to the <u>same scales as before</u> for p and ε, we have the means of comparing their tensile properties. Fig. 200 shows two other curves, representing the average behavior of steel and cast iron. At the respective elastic limits B, B′, and B″, it will be noticed that the curve for wrought iron makes a sudden turn from the vertical while those of the others curve away more gradually; that the curve for steel lies nearer the axis than the others which indicates a higher value for E_t; and that the ordinates BA, B′A′, and B″A″ (respectively 21000, 9000, and 30000 lbs. per sq inch) indicate the tensile stress at the elastic limit. These latter quantities will be called the moduli of tenacity at elastic limit for the respective materials.

Within the elastic limit the curves are nearly straight (proving Hooke's law) and if α, α', and α'' are the angles made

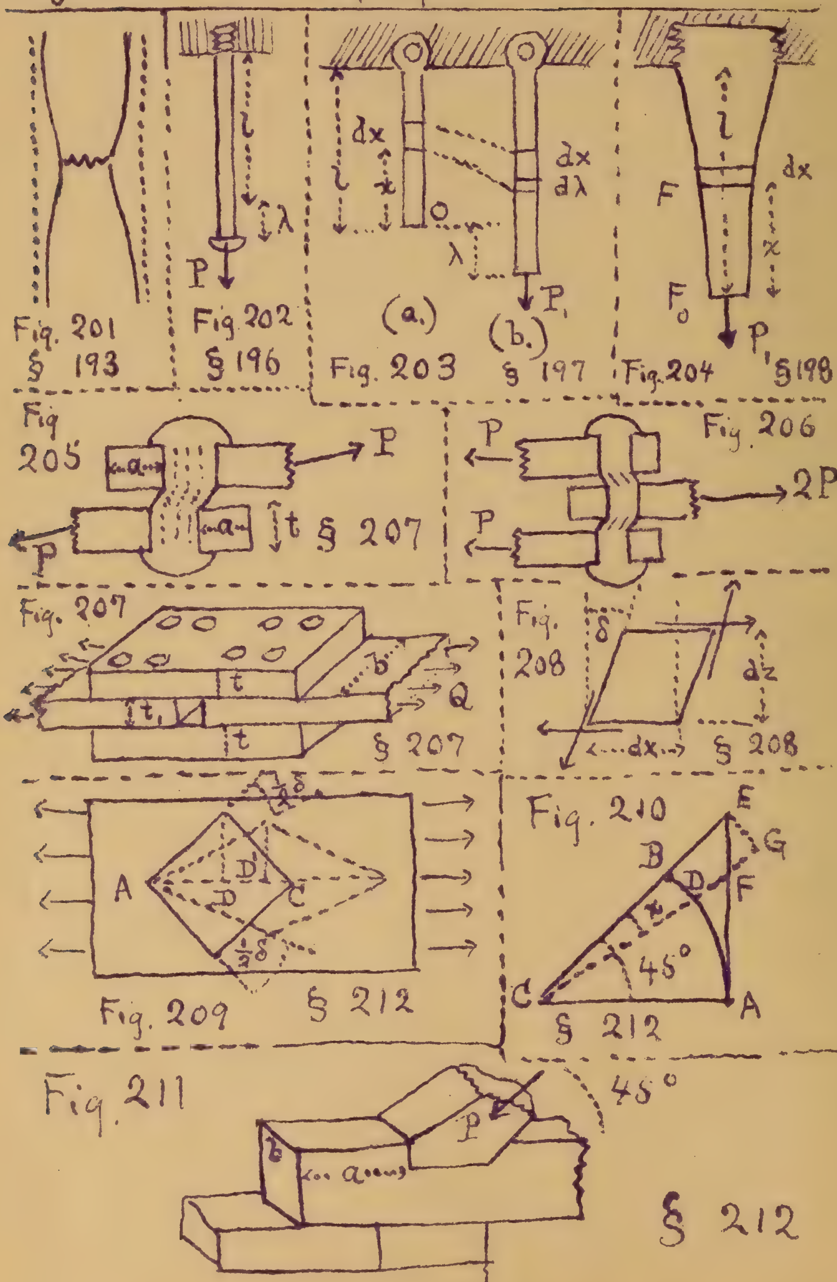
by these straight portions with the axis of X (i.e., of ε), we shall have

$$(E_1 \text{ for w. iron}) : (E_1 \text{ c. iron}) : (E_1 \text{ steel}) :: \tan\alpha : \tan\alpha' : \tan\alpha''$$

as a graphic relation between their moduli of elasticity; (since $E_1 = \frac{p}{\varepsilon}$).

Beyond the elastic limit, the wrought iron rod shows large increments of elongation for small increments of stress, i.e. the curve becomes nearly parallel to the horizontal axis, until rupture occurs at a stress of 53 000 lbs. per sq. inch of original sectional area (at rupture this area is somewhat reduced especially in the immediate neighborhood of the section of rupture; see next article) and after a relative elongation " ε = about 0.30 or 30 %. (The preceding table shows only a portion of the results) The curve for steel shows a much higher breaking stress (100 000 lbs. per sq. in.) than the wrought iron but the total elongation is smaller, ε = about 10 %. This is an average curve; tool steels give an elongation at rupture of about 4 or 5 %, while soft steels resemble wrought iron in their ductility, giving an extreme elongation of from 10 to 20 %. Their breaking stresses range from 70000 to 150 000 lbs. or more per sq. inch. Cast iron, being comparatively brittle, reaches at rupture an elongation of only 3 or 4 tenths of one per cent, the rupturing stress being about 18 000 lbs. per sq. inch. The elastic limit is rather ill defined in the case of this metal; and the proportion of carbon and the mode of manufacture have much influence on its behavior under test.

193. LATERAL CONTRACTION. In the stretching of prisms of nearly all kinds of material, accompanying the elongation of length is found also a diminution of width whose relative amount in the case of the three metals just treated is about $\frac{1}{3}$ or $\frac{1}{4}$ of the relative elongation (within elastic limit). Thus in the third ex-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periment in the table of §191, this relative lateral contraction or decrease of diameter $= \frac{1}{3}$ to $\frac{1}{4}$ of ε, i.e. about 0.00008. In the case of cast iron and hard steels this contraction is not noticeable except by very delicate measurements, both within and without the elastic limit; but the more ductile ~~metals as wrought~~ iron and the soft steels, when stretched beyond the elastic limit show this feature of their deformation in a very marked degree. Fig. 201 shows by dotted lines the original contour of a wrought iron rod while the continuous lines indicate that at rupture. At the cross section of rupture, whose position is determined by some local weakness, the drawing out is peculiarly pronounced. The contraction of area thus produced is sometimes as great as 50 or 60 % at the fracture.

194. "FLOW OF SOLIDS". When the change in relative position of the elements of a solid is extreme, as occurs in the making of lead pipe, drawing of wire, the stretching of a rod of ductile metal as in the preceding article, we have instances of what is called the flow of Solids, interesting experiments on which have been made by Tresca.

195. MODULI OF TENACITY. The tensile stress per square inch (of original sectional area) required to rupture a prism of a given material will be denoted by T and called the modulus of ultimate tenacity; similarly the modulus of safe tenacity, or greatest safe tensile stress on an element, by T'; while the tensile stress at elastic limit may be called T''. The ratio of T' to T'' is not fixed in practice but depends upon circumstances.

Hence if a prism of any material sustains a total pull or load P, and has a sectional area $= F$, we have

$$\left.\begin{array}{ll} P = FT & \text{for the ultimate or breaking load} \\ P' = FT' & \text{" " safe load} \\ P'' = FT'' & \text{" " load at elastic limit.} \end{array}\right\} \quad (2)$$

T' should always be less than T''.

196. RESILIENCE OF A STRETCHED PRISM. Fig. 202
In the gradual stretching of a prism, fixed at one extremity, the value of the tensile force P at the other necessarily depends on the elongation λ at each stage of the shortening, according to the relation

$$\lambda = \frac{Pl}{FE_t} \dots (\S\ 191) \dots \qquad (3)$$

within the elastic limit. (If we place a weight G on the flanges of the unstretched prism and then leave it to the action of gravity and the elastic action of the prism, the weight begins to sink, meeting an increasing pressure P, proportional to λ, from the flanges) Suppose the stretching to continue until P reaches some value P'' (at elastic limit say), and λ a value λ''. Then the work done so far is

$$W = \text{mean force} \times \text{space} = \tfrac{1}{2} P'' \lambda'' \qquad (4)$$

But from (2) $P'' = FT''$, and (see §§ 184 and 191)

$$\lambda = \varepsilon l$$

$\therefore$ (4) becomes $\quad W = \frac{1}{2} T''\varepsilon''. Fl = \frac{1}{2} T''\varepsilon'' V \qquad (5)$

where V is the volume of the prism. The quantity $\frac{1}{2}T''\varepsilon''$, or work done in stretching to the elastic limit a cubic inch of the given material, Weisbach calls the <u>Modulus</u> of <u>Resilience for tension</u>. From (5) it appears that the amounts of work done in stretching to the elastic limit prisms of the same material but of different dimensions are proportional to <u>their volumes</u> simply

The quantity $\frac{1}{2}T''\varepsilon''$ is graphically represented by the area of one of the triangles such as $OA'B$, $OA''B''$ in fig. 200; for (on the curve for wrought iron for instance) the modulus of tenacity at elastic limit is represented by $A'B$, and ε'' (i.e. ε for elastic limit) by OA'. The remainder of the area OBC included between the curve and the horizontal axis, i.e. from B to C, represents the work done in stretching a cubic inch from the elastic limit to the point of rupture; for each vertical strip having an altitude $= p$ and a width $= d\varepsilon$, has an area $= p d\varepsilon$

i.e. the work done by the stress p on one face of a cubic inch through the distance $d\varepsilon$, or increment of elongation.

If a weight or load $= G$ be "suddenly" applied to stretch the prism, i.e. placed on the flanges, barely touching them, and then allowed to fall, when it comes to rest again it has fallen through a height λ_1 and experiences at this instant some pressure P_1 from the flanges. $P_1 = ?$ The work $G\lambda'$ has been entirely expended in stretching the prism, none in changing the kinetic energy of G, which $= 0$ at both beginning and end of the distance λ_1,

$$\therefore G\lambda_1 = \tfrac{1}{2}P_1\lambda_1 \qquad \therefore \quad P_1 = 2G.$$

Since $P_1 = 2G$, i.e. is $> G$, the weight does not remain in this position but is pulled upward by the elasticity of the prism. In fact, the motion is <u>harmonic</u> (see §§ 59 and 138) Theoretically the elastic limit not being passed, the oscillations should continue indefinitely.

Hence a load G "suddenly applied" occasions double the tension it would if compelled to sink gradually by a support underneath, which is not removed until the tension is just $= G$. Oscillation is thus prevented.

If the weight G sinks through a height $= h$ before striking the flanges, fig. 202, we shall have similarly, within elastic limit, if $\lambda_1 =$ greatest elongation,

$$G(h+\lambda_1) = \tfrac{1}{2}P_1\lambda_1 \qquad (6)$$

If the elastic limit is to be just reached we have from eqs. (5) and (6), neglecting λ_1 compared with h,

$$Gh = \tfrac{1}{2}T''\varepsilon''.V \qquad (7)$$

an equation of condition that the prism shall not be injured.

197. STRETCHING OF A PRISM BY ITS OWN WEIGHT. In the case of a very long prism such as a mining-pump rod, its weight must be taken into account as well as that of the terminal load P_1; see fig. 203. At (a) the prism is shown in its unstrained condition, at (b) strained by the load P_1 and its own weight. Let the cross section be $= F$, the heaviness of the

prism $=\gamma$. Then the relative extension of any element at a distance x from O is

$$\varepsilon = \frac{d\lambda}{dx} = \frac{(P_1 + \gamma F x)}{FE_t} \qquad (1)$$

(See eq. (1) § 191); since $P_1 + F\gamma x$ is the load hanging upon the cross section at that locality. Equal dx's, therefore are unequally elongated, x varying from 0 to l. The total elongation is

$$\lambda = \int_0^l d\lambda = \frac{1}{FE_t}\int_0^l [P_1 dx + \gamma F x dx] = \frac{P_1 l}{FE_t} + \frac{\frac{1}{2}Gl}{FE_t}$$

I.e., $\lambda =$ the amount due to P_1, plus an extension which half its own weight would produce hung at the lower extremity.

The foregoing relates to the deformation of the piece, and is therefore a problem in stiffness. As to the strength of the prism, the relative elongation $\varepsilon = d\lambda \div dx$ [see eq. (1)], which is variable, must nowhere exceed a safe value $\varepsilon' = T' \div E_t$ (§§ 191 and 194). Now the greatest value of the ratio $d\lambda : dx$, by inspecting eq. (1), is seen to be at the upper end where $x = l$. The proper cross section F, for a given load P_1, is thus found.

Putting $\dfrac{P_1 + \gamma F x}{FE_t} = \dfrac{T'}{E_t}$, we have $F = \dfrac{P_1}{T' - \gamma l}$ (2)

198. SOLID OF UNIFORM STRENTH IN TENSION.

or hanging body of minimum material supporting its own weight and a terminal load P_1. Let it be a solid of revolution. If every cross-section F at a distance $= x$ from the lower extremity bears its safe load FT', every element of the body is doing full duty, and its form is the most economical of material.

The lowest section must have an area $F_0 = P_1 \div T'$, since P_1 is its safe load. Fig. 204. Consider any horizontal lamina; its weight is $\gamma F dx$, (γ = heaviness of the material, supposed homogeneous), and its lower base F must have $P_1 + G$ for its safe load, i.e.

$$G + P_1 = FT' \qquad (1)$$

in which G denotes the weight of the portion of the solid below F. Similarly for the upper base F+dF, we have

$$G + P_1 + \gamma F dx = (F+dF)T' \qquad (2)$$

By subtraction we obtain

$$\gamma F dx = T' dF; \qquad i.e. \frac{\gamma}{T'} dx = \frac{dF}{F}$$

in which the two variables x and F are separated. By integration we now have

$$\frac{\gamma}{T'}\int_0^x dx = \int_{F_0}^{F} \frac{dF}{F}; \text{ or } \frac{\gamma x}{T'} = \log_e \frac{F}{F_0} \qquad (3)$$

$$\text{i.e., } F = F_0 e^{\frac{\gamma x}{T'}} = \frac{P_1}{T'} e^{\frac{\gamma x}{T'}} \qquad (4)$$

from which F may be computed for any value of x.

The weight of the portion below any F is formed from (1) and (4); i.e.

$$G = P_1\left(e^{\frac{\gamma x}{T'}} - 1\right); \qquad (5)$$

while the total extension λ will be

$$\lambda = \varepsilon'' \frac{T'}{T''} l \qquad (6)$$

the relative elongation $d\lambda \div dx$ being the same for every dx, and bearing the same ratio to ε'' (at elastic limit), as T' does to T''.

199. TENSILE STRESSES INDUCED BY TEMPERATURE. If the two ends of a prism are immovably fixed, when under no strain and at a temperature t, and the temperature is then lowered to a value t', the body suffers a tension proportional to the fall in temperature (within elastic limit). If for a rise or fall of 1° Fahr. (or Cent.) a unit of length of the material would change in length by an amount η (called the coefficient of expansion) a length $= l$ would be contracted an amount $\lambda = \eta l(t-t')$ during the given fall of temperature if one end were free. Hence if this contraction is prevented by fixing both ends, the rod must be under a tension P, equal in value to the force which would be necessary to produce the elongation λ, just stated, under ordinary circumstances at the lower

temperature.

From eq.(1) §191, therefore, we have for this tension due to fall of temperature

$$P = \frac{E_t F}{l} \eta l (t - t') = E_t F (t - t') \eta$$

COMPRESSION OF SHORT BLOCKS.

200. SHORT AND LONG COLUMNS. In a prism in tension, its own weight being neglected, all the elements between the localities of application of the pair of external forces producing the stretching are in a state of stress, if the external forces act axially (excepting the few elements in the immediate neighborhood of the forces; these suffering local stresses dependent on the manner of application of the external forces); and the prism may be of any length without vitiating this statement. But if the two external forces are directed toward each other the intervening elements will not all be in the same state of compressive stress unless the prism is comparatively short, (or unless numerous points of lateral support are provided) A long prism will buckle out sideways, thus even inducing tensile stress, in some cases, in the elements on the convex side.

Hence the distinction between short blocks and long columns. Under compression the former yield by crushing or splitting, while the latter give way by flexure (i.e. bending.) Long columns, then will be treated separately in a subsequent chapter. In the present section the blocks treated being about three or four times as long as wide, all the elements will be considered as being under equal compressive stresses at the same time.

201. NOTATION FOR COMPRESSION. By using another subscript, we have for compression

Modulus of Elasticity $= E_c$;

while we may write

C for the Modulus of Crushing,

C' for " " of safe compression, and

C'' for " " of compression at elastic limit.

For the absolute and relative shortening in length we may still

use λ and ε, respectively, and within the elastic limit may write equations similar to those for tension, F being the sectional area of the block and P one of the terminal forces (see fig. 205), while p = compressive stress per square inch of F, viz.:

$$E_c = \frac{p}{\varepsilon} = \frac{P \div F}{d\lambda \div dx} = \frac{P \div F}{\lambda \div l} = \frac{Pl}{F\lambda} \qquad (1)$$

within the elastic limit.

Also for a short block

$$\left.\begin{aligned} &\text{Crushing force} = FC \\ &\text{Compressive force at elastic limit} = FC'' \\ &\text{Safe compressive force} = FC'. \end{aligned}\right\} \qquad (2)$$

202. REMARKS ON CRUSHING. As in §182 for a tensile stress, so for a compressive stress we may prove that a shearing stress $= p \sin\alpha \cos\alpha$ is produced on planes at an angle α with the axis of the short block, p being the compression per unit of area of transverse section. Accordingly it is found that short blocks of many comparatively brittle materials yield by shearing on planes making an angle of about 45° with the axis, the expression $p \sin\alpha \cos\alpha$ reaching a maximum for $\alpha = 45°$; that is, wedge-shaped pieces are forced out from the sides. Hence the necessity of making the block three or four times as long as wide, since otherwise the friction on the ends would cause the piece to show a greater resistance by hindering this lateral motion. Crushing by splitting into pieces parallel to the axis sometimes occurs.

Blocks of ductile materials, however, yield by swelling out, or bulging, laterally, resembling plastic bodies somewhat in this respect.

The elastic limit is more difficult to locate than in tension, but seems to have about the same position in comparison as in tension, in the case of wrought iron and steel. For each of the same metals and for cast iron it is also found that $E_c = E_t$ quite nearly, so that the single symbol E may be used for both.

EXAMPLES IN TENSION & COMPRESSION.

203. TABLES FOR TENSION AND COMPRESSION. The round numbers of the following table are to be taken as

TABLE of the MODULI etc. of materials in TENSION.

Material	ε'' (Elast. lim.) abst. number	ε At Rupture abs. number	E_t Modul. of Elast. lbs. per sq. inch	T'' Elast. lim. lbs. per sq. in.	T ultimate lbs. per sq. in.
Soft Steel	.00200	.2500	26 000 000	50000	80000
Hard Steel	.00200	.0500	40 000 000	90 000	130000
Cast Iron	.00066	.0020	14 000 000	9 000	18000
Wro't Iron	.00080	.2500	28 000 000	22 000	60000
Brass	.00100		10 000 000	7000 to 19000	16000 to 50000
Glass	- - -	- - -	9 000 000	- - -	3500
Wood, with the fibres,	.00200 to .01100	.0070 to .0150	200 000 to 2000 000	3000 to 19000	6000 to 28000
Hemp rope	- - -	- - -	- - -	- - -	7000

averages only, for use in the numerical examples following. (The scope and design of the present work admit of nothing more. For abundant detail of the results of the more important experiments of late years the student is referred to the recent works of Profs. Thurston, Burr, Lanza, and Wood) Another column might have been added giving the Modulus of Resilience in each case, viz.: $\frac{1}{2}\varepsilon''T''$ (which also $= \frac{T''^2}{E_t}$); see §195. ε is an abstract number, and $= \lambda \div l$, while

E_t, T', and T, are given in pounds per square inch.

TABLE of MODULI, etc. COMPRESSION of SHORT BLOCKS.

Material.	ϵ'_c elast. limit abst. numb.	ϵ_c at rupture abs. numb.	E_c Mod. of Elas. lbs. per sq. in.	C' elastic lim. lbs. per sq. in.	C rupture. lbs. per sq. in.
Soft Steel	0.00100		30 000 000	30 000	
Hard Steel	0.00120	0.3000	40 000 000	50 000	200 000
Cast Iron	0.00150		14 000 000	20 000	90 000
Wrought "	0.00080	0.3000	28 000 000	24 000	50 000
Glass					20 000
Granite					10 000
Sandstone					5 000
Brick					3 000
Wood {with the fibres}		0.0100 to 0.0400	350 000 to 2 000 000		2 000 to 10 000
Portland Cement					4 000

204. EXAMPLES. No. 1 A bar of tool steel, of sectional area = 0.097 sq. inches, is ruptured by a tensile force of 14000 lbs. A portion of its length, originally $\frac{1}{2}$ a foot, is now found to have a length of 0.532 ft. Required T, and ϵ at rupture. Using the inch and pound as units (as in the foregoing tables) we have

$$T = \frac{14000}{.097} = 144000 \text{ lbs. per sq. in.,} \quad \text{while}$$

$\varepsilon = (0.532 - 0.5) \times 12 \div (0.50 \times 12) = 0.064$

Example 2. Tensile test of a bar of "Hay Steel" for the Glasgow Bridge, Missouri. The portion measured was originally 3.21 ft. long and 2.09 in. X 1.10 in. in section. At the elastic limit P was 124200 lbs., and the elongation was 0.064 ins. Required E_t, T'', and ε'' (for elastic limit.)

$$\varepsilon = \frac{\lambda}{l} = \frac{0.064}{3.21 \times 12} = .00165 \text{ at elastic limit.}$$

$$T'' = 124200 \div (2.09 \times 1.10) = 54000 \text{ lbs. per sq. in.}$$

$$E_t = \frac{p}{\varepsilon} = \frac{P}{F\varepsilon} = \frac{124200}{230 \times .00165} = 32\ 570\ 000$$

lbs. per sq. inch. Nearly the same result for E_t would probably have been obtained for values of p and ε below the elastic limit.

The Modulus of Resilience of the above steel (see §196) would be $\frac{1}{2}\varepsilon'' T'' = 44.82$ inch-pounds of work per cubic inch of metal, so that the whole work expended in stretching to the elastic limit the portion above cited is

$$W = \tfrac{1}{2}\varepsilon'' T'' V = 3968. \text{ inch-lbs.}$$

An equal amount of work will be done by the rod in recovering its original length.

Example 3. A hard steel rod of $\frac{1}{2}$ sq. in. section and 20 ft. long is under no stress at a temperature of 130° Cent., and is provided with flanges so that the slightest contraction of length will tend to bring two walls nearer together. If the resistance to this motion is 10 tons how low must the temperature fall to cause any motion? η being $= .0000120$, (Cent. scale) From §199, we have, expressing P in lbs. and F in sq. inches, since $E_t = 40\ 000\ 000$ lbs. per sq. inch.

$$10 \times 2000 = 40\ 000\ 000 \times \tfrac{1}{2} \times (130 - t') \times 0.000012$$

whence $t' = 46.6°$ Centigrade.

Example 4. If the ends of an iron beam bearing 5 tons at its middle rest upon stone piers, required the necessary bearing surface, putting C', for stone, $= 200$ lbs. per sq. inch.

Example 5. How long must a wrought iron rod be, supported vertically at its upper end, to break with its own weight?

Example 6. One voussoir of an arch-ring presses its neighbor with a force of 50 tons, the joint having a surface of 5 sq. feet; required the compression per sq. inch.

205. FACTOR OF SAFETY. When, as in the case of stone, the value of the stress at the elastic limit is of very uncertain determination by experiment, it is customary to refer the value of the safe stress to that of the ultimate by making it the nth portion of the latter. n is called a factor of safety, and should be taken large enough to make the safe stress come within the elastic limit. For stone n should not be less than 10, i.e. $C' = C \div n$; (see Ex. 6 just given).

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206. PRACTICAL NOTES. It was discovered independently by Commander Beardslee and Prof. Thurston, in 1873, that if wrought iron rods were strained considerably beyond the elastic limit and allowed to remain free from strain for at least one day thereafter, a second test would show higher limits both elastic and ultimate.

When articles of cast iron are imbedded in oxide of iron and subjected to a red heat for some days, the metal loses most of its carbon, and is thus nearly converted into wrought iron, lacking, however, the property of welding. Being malleable, it is called malleable cast iron.

Chrome steel (iron and chromium) and tungsten steel possess peculiar hardness, fitting them for cutting tools, rock drills, picks, etc.

By fatigue of metals we understand the fact, recently discovered by Wöhler in experiments made for the Prussian Government, that rupture may be produced by causing the stress on the elements to vary repeatedly between two limiting values the higher of which may be considerably below T (or C), the number of repetitions necessary to produce rupture being dependent both on the range of variation and the higher value.

For example, in the case of Phoenix iron in tension, rupture was produced by causing the stress to vary from 0 to 52800 lbs. per sq inch, 800 times; also, from 0 to 44000 lbs. per sq inch

340853 times; while 4 000 000 variations between 26400 to 48400 per sq. inch did not cause rupture. Many other experiments were made and the following conclusions drawn (among others):

Unlimited repetitions of variations of stress (lbs. per sq. in.) between the limits given below will not injure the metal (Prof. Burr's Materials of Engineering)

Wrought iron	From 17600 Comp. to 17600 Tension
	" 0 to 33000 "
Axle Cast Steel	From 30800 Comp. to 30800 Tension.
	" 0 to 52800 "
	" 38500 Tens. to 88000 "

SHEARING

207. RIVETS. The angular distortion called shearing strain, in the elements of a body is specially to be provided for in the case of <u>rivets</u> joining two or more plates. This distortion is shown, in figs. 205 and 206, in the elements near the plane of contact of the plates, much exaggerated. In fig. 205 (a lap-joint) the rivet is said to be in <u>single shear</u>; in fig. 206 in <u>double shear</u>. If P is just great enough to shear off the rivet, the <u>modulus of ultimate shearing</u>, which may be called S, will be

$$S = \frac{P}{F} = \frac{P}{\frac{1}{4}\pi d^2} \qquad (1)$$

in which F = the cross section of the rivet, its diameter being = d. For safety a value $S' = \frac{1}{4}$ to $\frac{1}{6}$ of S should be taken for metal, in order to be within the elastic limit.

As the width of the plate is diminished by the rivet hole the remaining sectional area of the plate should be ample to sustain the tension P′, or 2P′, (according to the plate considered) P′ being the safe shearing force for the rivet. Also the thickness t of the plate should be such that the side of the hole shall be secure against crushing; P′ must be > C′td.

Again, the distance a, fig. 205, should be sufficient to prevent

the tearing or shearing out of the part of the plate between the rivet and edge of the plate.

For economy of material the seam or joint should be no more liable to rupture by one than by another, of the four modes just mentioned. The relations which must then subsist will be illustrated in the case of the "butt-joint" with two cover-plates, Fig. 207. Let the dimensions be denoted as in the figure and the total tensile force on the joint be $= Q$. Each rivet (see also Fig 206) is exposed in each of two of its sections to a shear of $\frac{1}{8}Q$, hence for safety against shearing of rivets we put

$$\frac{1}{8}Q = \frac{1}{4}\pi d^2 S' \quad \ldots\ldots\ldots (1)$$

Along one row of rivets in the main plate the sectional area for resisting tension is reduced to $(b-2d)t_1$, hence for safety against rupture of that plate by the tension Q, we put

$$Q = (b-2d)t_1 T' \quad \ldots\ldots (2)$$

Equations (1) and (2) suffice to determine d for the rivets and and t_1 for the main plates, Q and b being given; but the values thus obtained should also be examined with reference to the compression in the side of the rivet hole, i.e. $C' t_1 d$ must be $< \frac{1}{4}Q$. [The distance a, Fig 205, to the edge of the plate is recommended by different authorities to be from d to $3d$.]

Similarly for the cover-plate we must have $\frac{1}{2}Q \overset{=}{\underset{<}{\text{or}}} (b-2d)tT'$ and $C' t d < \frac{1}{8}Q$

If the rivets do not fill their holes closely a large margin should be allowed in practice. Again, in boiler work, the pitch, or distance between centers of two consecutive rivets may need to be smaller, to make the joint steam-tight, than would be required for strength alone.

208. SHEARING DISTORTION. The change of form in an element due to shearing is an angular deformation and will be measured in π-measure. This angular change, or differ

ence between the value of the corner angle during strain and $\frac{1}{2}\pi$, its value before strain, will be called δ, and is proportional (within elastic limit) to the shearing stress per unit of area, p_s, existing on all the four faces whose angles with each other have been changed; Fig. 208. (See § 181) By § 184 the MODULUS OF SHEARING ELASTICITY is the quotient obtained by dividing p_s by δ; i.e.

$$E_s = \frac{p_s}{\delta}$$

or inversely, $\delta = p_s \div E_s$.

The value of E_s for different substances is most easily determined by experiments on torsion in which shearing is the most prominent stress. (This prominence depends on the position of the bounding planes of the element considered; e.g., in fig. 208, if another element were considered within the one there shown and with its planes at 45° with those of the first, we should find tension alone on one pair of opposite faces, compression alone on the other pair) It will be noticed that shearing stress cannot exist on two opposite faces only, but also on the other two, forming a couple of equal and opposite moment to the first, this being necessary for the equilibrium of the element, even when tensile or compressive stresses are also present on the same faces.

209. SHEARING STRESS IS ALWAYS OF THE SAME INTENSITY ON THE FOUR FACES OF AN ELEMENT. (By intensity is meant per unit of area; and the four faces referred to are those perpendicular to the paper in fig. 208, the shearing stress being parallel to the paper.)

Let dx and dz be the width and height of the element in fig. 208, while dy is its thickness perpendicular to the paper. Let the intensity of the shear on the right hand face be $= q_s$ that on the top face $= p_s$. Then for the element as a free body, taking moments about the axis O perpendicular to paper, we have

$$q_s\, dz\, dy \times dx - p_s\, dx\, dy \times dz = 0 \therefore q_s = p_s$$

Even if there were also tensions (or compressions) on one or both pairs of faces their moments about O would balance (or fail to do so by a differential of a higher order) independently of the shears, and the above result would still hold.

210. TABLE OF MODULI FOR SHEARING.

Material	δ'' i.e. δ at elastic limit	E_s	S'' (elast. limit)	S (rupture)
	arc in π-measure	lbs. per sq. in.	lbs. sq. in.	lbs. sq. in.
Soft Steel		9 000 000		70 000
Hard Steel	0.0032	14 000 000	45.000	90 000
Cast Iron	0.0021	7 000 000	15 000	30 000
Wro't Iron	0.0022	9 000 000	20 000	50 000
Brass		5 000 000		
Glass				
Wood, across fibre				1500 to 8000
Wood, along fibre				500 to 1200

As in the tables for tension and compression the above values are averages. The true values may differ from these as much as 30 per cent. in particular cases according to the quality of the specimen.

211. PUNCHING rivet holes in plates of metal requires the overcoming of the shearing resistance along the convex surface of the cylinder punched out. Hence if d = diameter of hole, and t = the thickness of the plate, the necessary force for the punching, the surface sheared being $F = t\pi d$, is

$$P = St\pi d$$

212. E AND E_s; THEORETICAL RELATION. In case a rod is in tension within the elastic limit the relative (linear) lateral contraction (let this $=m$) is so connected with E_t and E_s that if two of the three are known the third can be deduced theoretically. This relation is proved as follows, by Prof. Burr. Taking an elemental cube with four of its faces at 45° with the axis of the piece, fig. 209, the axial half-diagonal AD becomes of a length $AD' = AD + \varepsilon . AD$ under stress, while the transverse half diagonal contracts to a length $B'D' = AD - m . AD$. The angular distortion δ is supposed very small compared with 90° and is due to the shear p_s per unit of area on the face BC (or BA). From the figure we have

$$\tan\left(45^\circ - \frac{\delta}{2}\right) = \frac{B'D'}{AD'} = \frac{1+\varepsilon}{1-m} = 1 - m - \varepsilon, \text{ approx.}$$

[But, fig. 210, $\tan(45^\circ - x) = 1 - 2x$ nearly, when x is small; for, taking $CA = \text{unity} = AE$, $\tan AD = AF = AE - EF$. Now approximately $EF = EG\sqrt{2}$ and $EG = BD\sqrt{2} = x\sqrt{2}$ $\therefore AF = 1 - 2x$ nearly.] Hence

$$1 - \delta = 1 - m - \varepsilon; \text{ or } \delta = m + \varepsilon \qquad (2)$$

Eq. (2) holds good whatever the stresses producing the deformation, but in the present case of a rod in tension, if it is an isotrope, and if p = tension per unit of area on its transverse section, (see § 181 putting $\alpha = 45°$), we have $E_t = p \div \varepsilon$ and $E_s = (p_s \text{ on BC}) \div \delta = \frac{1}{2} p \div \delta$. Putting also $(m \div \varepsilon) = r$, whence $m = r\varepsilon$, eq. (2) may finally be written

$$\frac{1}{2E_s} = (r+1)\frac{1}{E_t}; \text{ i.e., } E_s = \frac{E_t}{2(1+r)} \qquad (3)$$

Prof. Bauschinger, experimenting with cast-iron rods, found that in tension the ratio $m : \varepsilon$ was $= \frac{23}{100}$, as an average, which in eq. (3) gives

$$E_s = \frac{100}{246} E_t = \frac{2}{5} E_t \text{ nearly.} \qquad (4)$$

His experiments on the torsion of cast iron rods gave $E_s = 6\,000\,000$ to $7\,000\,000$ lbs. per sq. inch. By (4), then, E_t should be $15\,000\,000$ to $17\,500\,000$ which is approximately true (§ 203)

Corresponding results may be obtained for short blocks in compression, the lateral change being a dilatation instead of a contraction.

213. EXAMPLES IN SHEARING.

Example 1. Required the proper length, a, Fig. 211, to guard against the shearing off, along the grain, of the portion, ab, of a wooden tie-rod, the force P being = 2 tons, and the width of the tie = 4 inches. Using a value of $S' = 100$ lbs. per sq. inch, we put $baS' = 4000 \cos 45°$; i.e. $a = (4000 \times 0.707) \div (4 \times 100) = 7.07$ inches

Example 2. A $\frac{7}{8}$ in. rivet of wrought iron, in single shear (see fig. 205) has an ultimate shearing strength

$$P = FS = \frac{1}{4}\pi d^2 S = \frac{1}{4}\pi\left(\frac{7}{8}\right)^2 \times 50000 = 30050 \text{ lbs.}$$

For safety, putting $S' = 8000$ instead of S, $P' = 4800$ lbs. is its safe shearing strength in single shear.

The wrought iron plate, to be secure against the side-crushing in the hole, should have a thickness t, computed thus:

$$P' = tdC'; \text{ or } 4800 = t \cdot \frac{7}{8} \cdot 12000 \therefore t = 0.46 \text{ in.}$$

If the plate were only 0.23 in. thick the safe value of P would be only $\frac{1}{2}$ of 4800.

Example 3. Conversely, given a lap-joint in which the plates are $\frac{1}{4}$ in. thick and the tensile force on the joint = 600 lbs. per linear inch of seam, how closely must $\frac{3}{4}$ inch rivets be spaced in one row, putting $S' = 8000$ and $C' = 12000$ lbs. per sq. in. Let the distance between centres of rivets be $= x$ (in inches), then the force upon each rivet = 0.44 sq. inches. Having regard to the shear

ing strength of the rivet we put $600x = 0.44 \times 8000$ and obtain $x = 4.20$ in.; but considering that the safe crushing resistance of the ~~hole~~ is $= \frac{1}{4} \cdot \frac{3}{4} \cdot 12000 =$ 2250 lbs., $600x = 2250$ gives $x = 3.75$ inches, which is the pitch to be adopted. What is the tensile strength of the reduced sectional area of the plate, with this pitch?

Example 4. Butt joint; (see fig. 207); $\frac{3}{8}$ in. plate; $\frac{3}{4}$ in. rivets; $T' = C' = 12000$; $S' = 8333$; width of plates = 14 inches. Will one row of rivets be sufficient at each side of joint, if $Q = 30\,000$ lbs.? The number of rivets = ? Here each rivet is in double shear and has therefore a double strength as regards shear. In double shear the safe strength of each rivet $= 2FS' = 3260$ lbs. Now $30000 \div 3260 = 9.2$, say 9.0. With the nine rivets in one row the reduced section of the inside plate = $(14^{in} - 9 \times \frac{3}{4}^{in} = 7.25 \text{ in.}) \times \frac{3}{8} = 2.72$ sq. in. whose safe tensile strength $= 2.72 \times 12000 = 32640$ lbs. which is > 30000 lbs. As for side crushing, $\frac{1}{9}$ of $30000 = 3333$. lbs. which is less than $(\frac{3}{8} \cdot \frac{3}{4} \times 12000 =) 3375$ lbs.

Hence nine rivets in one row are sufficient.

Chap. II. Torsion.

214. ANGLE OF TORSION AND OF HELIX. When a cylindrical beam or shaft is subjected to a twisting or torsional action, i.e. when it is the means of holding in equilibrium two couples in parallel planes and of equal and opposite moments, the longitudinal axis of symmetry remains straight and the elements along it experience no stress (whence it may be called the "line of no twist") while the lines originally parallel to it assume the form of helices each element of which is distorted in its angles (origin-

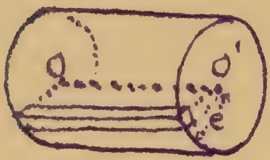


Fig. 212
§ 214

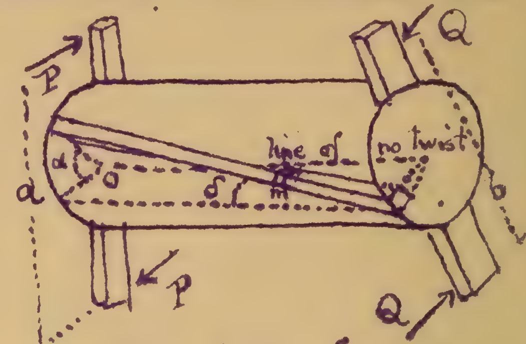


Fig. 213 § 212

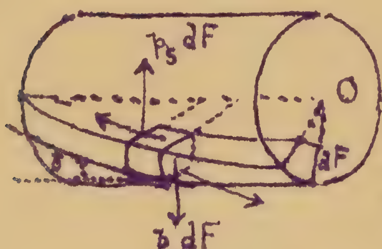


Fig. 214 § 215

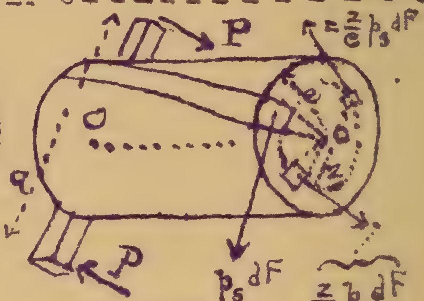


Fig. 215

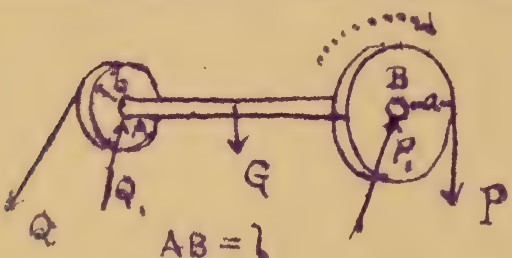


Fig. 216

§ 221

ally right angles) the amount of distortion being assumed proportional to the radius of the helix. The directions of the faces of any element were originally as follows: two radial, two in consecutive transverse sections, and the other two tangent to two consecutive circular cylinders whose common axis is that of the shaft. E.g. in Fig. 212 we have an unstrained shaft, while in Fig. 213 it holds the two couples (of equal moment $Pa = Qb$) in equilibrium. These couples act in parallel planes perpendicular to the axis of the prism and a distance, l, apart. Any surface element, m, which in fig. 212 was entirely right-angled is now distorted. Two of its angles have been increased, two diminished by an amount δ, the angle between the helix and a line parallel to the axis. Supposing m to be the most distant of any element from the axis, this distance being e, any other element at a distance z from the axis experiences an angular distortion $= \frac{z}{e}\delta$.

If now we draw OB' parallel to OA, the angle $BOB' = \alpha$, is called the ANGLE OF TORSION, while δ may be called the helix angle; the former lies in a transverse plane, the latter in a plane tangent to the cylinder. Now $\tan\delta = (\text{arc } BB') \div l$; but arc $BB' = e\alpha$; hence, putting δ for $\tan\delta$,

$$\delta = \frac{e\alpha}{l} \quad \ldots\ldots\ldots \quad (1)$$

215. SHEARING STRESS ON THE ELEMENTS. The angular distortion, or shearing strain, δ, of any element (bounded as already described) is due to the shearing stresses exerted on it by its neighbors on the four faces perpendicular to the tangent plane of the cylindrical shell in which the element is situated. Consider these neighboring elements of an outside element removed, and the stresses put in; the latter are accountable for the distortion of the element and so hold it in equilibrium. Fig. 214 shows this element "free". Within the elastic limit

δ is known to be proportional to p_s the shearing stress per unit of area on the faces whose relative angular positions have been changed. That is, from eq. (1) §208, $\delta = p_s \div E_s$; whence, see (1) of §214,

$$p_s = \frac{e\alpha E_s}{l} \qquad (2)$$

In (2), p_s and e both refer to a surface element, e being the radius of the cylinder, and p_s the greatest intensity of shearing stress existing in the shaft. Elements lying nearer the axis suffer shearing stresses of less intensity in proportion to their radial distances, i.e., to their helix-angles. That is, the shearing stress on that face of an element which forms a part of a transverse section and whose distance from the axis is z, is $p_, = \frac{z}{e} p_s$, per unit of area and the total shear on the face is $p dF$, dF being the area of the face.

216. TORSIONAL STRENGTH. We are now ready to expose the full transverse section of a shaft under torsion, to deduce formulae of practical utility. Making a right section of the shaft of fig. 213 anywhere between the two couples and considering the left-hand portion as a free body, the forces holding it in equilibrium are the two forces P of the left-hand couple and an infinite number of shearing forces, each tangent to its circle of radius z, in the cross section exposed by the removal of the right-hand portion. The cross section is assumed to remain plane during torsion, and is composed of an infinite number of dF's, each being an exposed face of an element; see fig. 215.

Each elementary shearing force $= \frac{z}{e} p_s dF$ and z is its lever arm about the axis OO'. For equilibrium, Σ (mom.) about the axis OO' must $= 0$; i.e. in detail

$$-P\tfrac{1}{2}a - P\tfrac{1}{2}a + \int \left(\tfrac{z}{e} p_s dF\right) z = 0$$

or, reducing,

$$\frac{p_s}{e} \int z^2 dF = Pa; \text{ or, } \frac{p_s I_p}{e} = Pa \qquad (3)$$

Eq. (3) relates to <u>torsional strength</u> since it contains p_s the greatest shearing stress induced by the torsional couple, whose moment Pa is called the MOMENT OF TORSION, the stresses in the cross

section forming a couple of equal and opposite moment.

I_p is recognized as the POLAR MOMENT OF INERTIA of the cross section, discussed in §94; e is the radial distance of the outermost element, and = the radius for a circular shaft.

217. TORSIONAL STIFFNESS. In problems involving the angle of torsion, or deformation of the shaft, we need an equation connecting Pa and α, which is obtained by substituting in eq. (1) the value of p_s in eq. (2), whence

$$\frac{\alpha I_p E_s}{l} = Pa \qquad (4)$$

From this it appears that the angle of torsion α is proportional to the moment of torsion, Pa, within the elastic limit; α must be expressed in π-measure. Trautwine cites 1° (i.e. $\alpha = 0.0174$) as a maximum allowable value for shafts.

218. TORSIONAL RESILIENCE is the work done in twisting a shaft from a unstrained state until the elastic limit is reached in the outermost elements. If in fig. 213 we imagine the right-hand extremity to be fixed, while the other end is gradually twisted through an angle α_1 each force P of the couple must be made to increase gradually from a zero value up to the value P_1 corresponding to α_1. In this motion each end of the arm a describes a space $= \frac{1}{2} a\alpha_1$, and the mean value of the force $= \frac{1}{2} P_1$ (see §17). Hence the work done in twisting is

$$U_1 = \frac{1}{2} P_1 \times \frac{1}{2} a\alpha_1 \times 2 = \frac{1}{2} P_1 a\alpha_1 \qquad (5)$$

By the aid of preceding equations (5) can be written.

$$U_1 = \frac{\alpha_1^2 E_s I_p}{2l} = \frac{P_1^2 a^2 l}{2 I_p E_s} = \frac{p_s^2 I_p l}{2 E_s e^2} \qquad (6)$$

If for p_s we write S'' (Modulus of safe shearing) we have for the resilience of the shaft.

$$U'' = \frac{S''^2 I_p l}{2 E_s e^2} \qquad (7)$$

If the torsional elasticity of an originally unstrained shaft is to be the means of arresting the motion of a moving mass whose weight is G, we write $U'' = \frac{G}{g} \cdot \frac{v^2}{2}$; as the condition that the shaft shall

not be injured. Here v is the initial velocity of the moving body.

219. POLAR MOMENT OF INERTIA. For a shaft of circular cross section (see §94) $I_p = \frac{1}{2}\pi r^4$; for a hollow cylinder $I_p = \frac{1}{2}\pi(r_1^4 - r_2^4)$; while for a square shaft $I_p = \frac{1}{6}b^4$, b being the side of the square; for a rectangular cross-section sides b and h, $I_p = \frac{1}{12}bh(b^2+h^2)$.

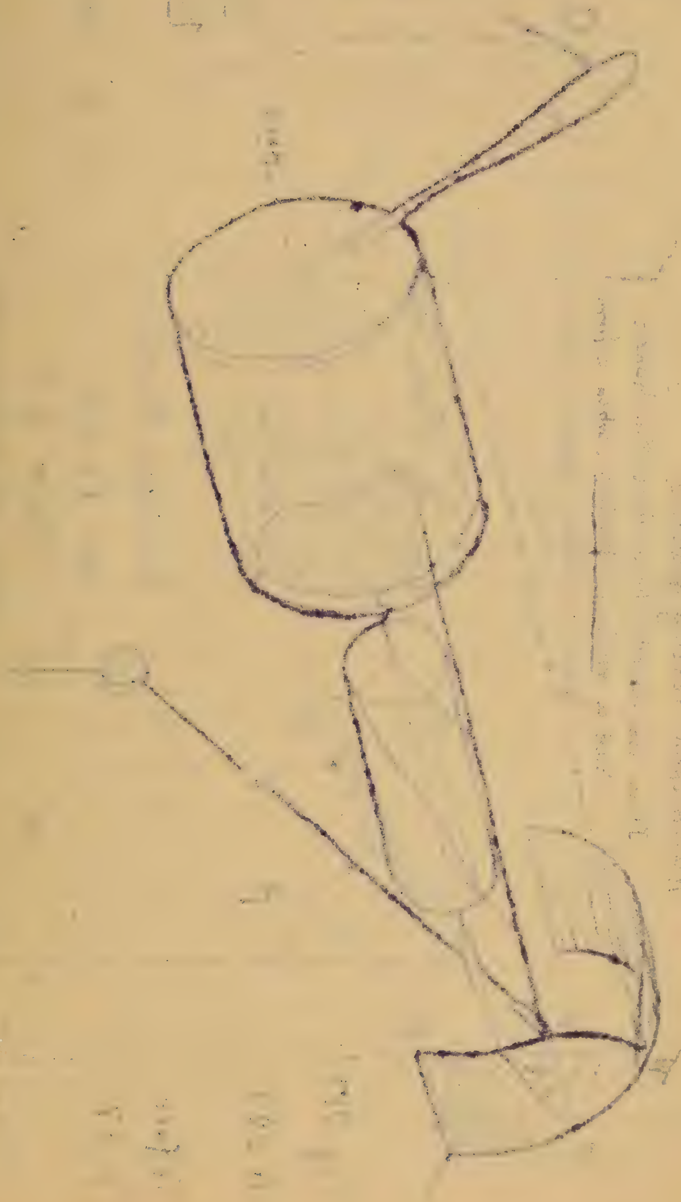
220. NON-CIRCULAR SHAFTS. If the cross-section is not circular it becomes warped in torsion, instead of remaining plane. Hence the fore-going theory does not strictly apply. The celebrated investigations of St. Venant however cover many of these cases. (See § 708 of Thompson and Tait's Natural Philosophy; also Prof. Burr's Elasticity and Strength of the Materials of Engineering) His results give for a square shaft (instead of the $\frac{\alpha b^4 E_s}{6l} = Pa$ of eq. (4) of § 217)

$$0.841\frac{\alpha b^4 E_s}{6l} = Pa$$

and $Pa = \frac{1}{5}b^3 p_s$, instead of eq. (3) of § 216, p_s being the greatest shearing stress.

The elements of greatest shearing strain are found at the middles of the sides, instead of at the corners, when the prism is of square or rectangular cross-section. The warping of the cross-section in such a case is easily verified by the student in twisting a bar of india-rubber in his fingers.

221. TRANSMISSION OF POWER. Fig. 216. Suppose the cog-wheel A to cause B, on the same shaft, to revolve uniformly and overcome a resistance Q, the pressure of the teeth of another cog-wheel; A being driven by still another wheel. The shaft AB is under torsion, the moment of torsion being $= Pa = Qb$. (P_1 and Q_1 the bearing reactions have no moment about the axis of the shaft.) If the shaft makes u revolutions per minute, the work transmitted (transmitted; not



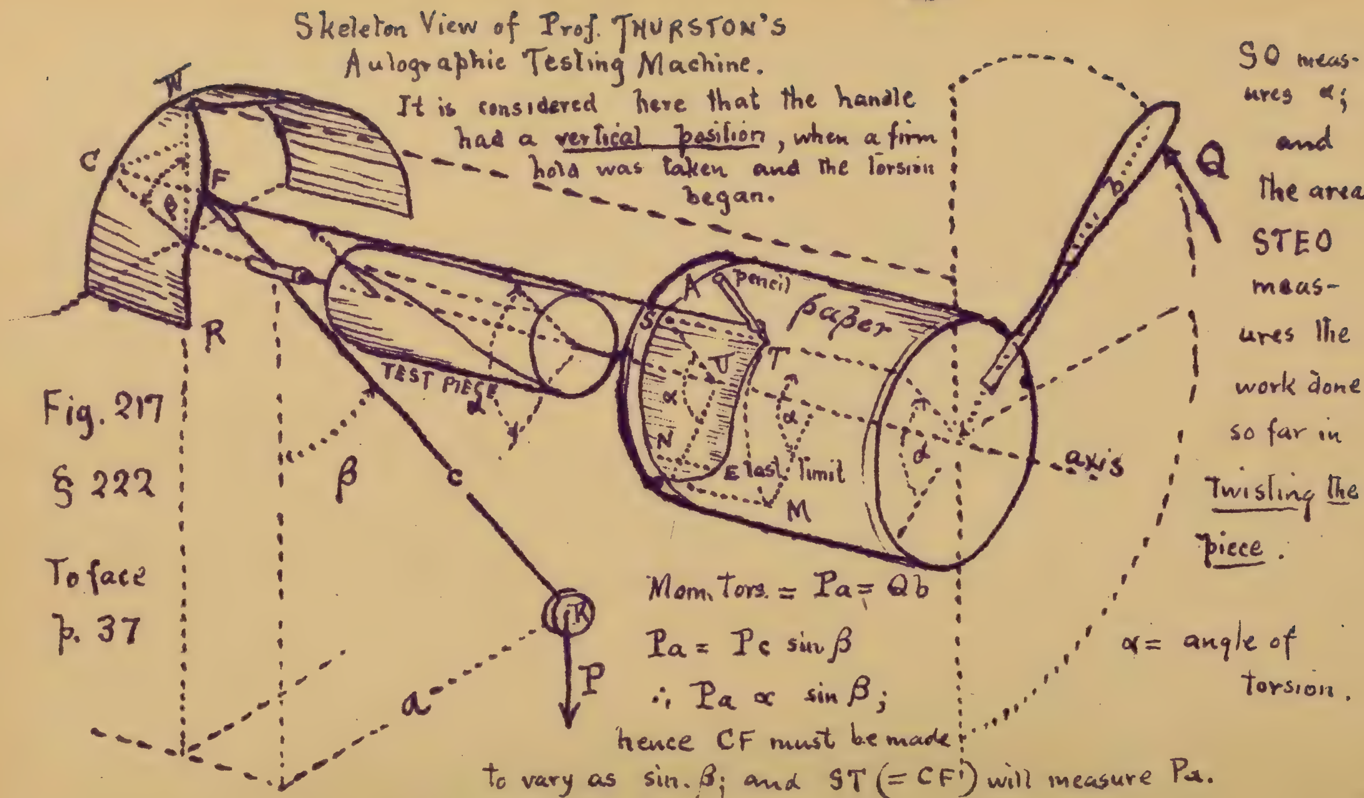


Fig. 217
§ 222
To face p. 37

expended in twisting the shaft whose angle of torsion remains constant, corresponding to Pa) per minute, i.e. the POWER is

$$L = P \cdot 2\pi a \cdot u = 2\pi u Pa \qquad (8)$$

Now in the Strength of Materials the pound and inch are the units of the tables for S, C, etc.; hence to reduce L to Horse Power, = (H.P.), we divide by 33 000 × 12, i.e. the number of inch-pounds per minute constituting one H.P.,

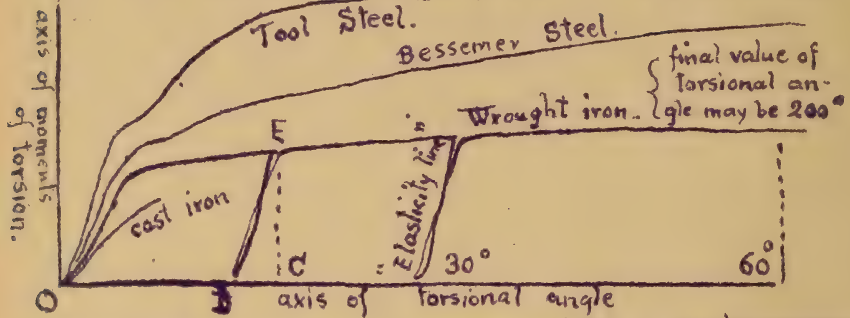
$$\therefore \text{H.P. transmitted} = \frac{1}{6} \cdot \frac{\pi u Pa}{33000} \quad \text{---} \begin{cases}\text{inch}\\ \text{pound}\\ \text{minute}\end{cases}$$

For the safety of the shaft Pa must $= \frac{S' I_p}{e}$, from eq. (3), and with $I_p = \frac{1}{2}\pi r^4$ and $e = r$, we have finally, solving for r

$$r = \sqrt[3]{\frac{396\,000 (\text{H.P.})}{\pi^2 S' u}} \quad \text{---} \begin{cases}\text{inch}\\ \text{pound}\\ \text{minute}\end{cases} \quad \text{---} \qquad (9)$$

S' may be made 7000 lbs. per sq. inch for wrought-iron; 10 000 for steel, and 5000 for cast-iron. Eq. (9) will then give r in INCHES. If the value of Pa fluctuates periodically, as when a shaft is driven by a connecting rod and crank, the radius of the shaft must be made greater than the above value in the ratio of $\sqrt[3]{m} : 1$, m being the ratio of the maximum to the mean torsional moment; $m =$ about $1\frac{1}{2}$ under ordinary circumstances (Cotterill).

222. AUTOGRAPHIC TESTING MACHINE. The principle of Prof. Thurston's invention bearing this name is shown in fig. 217. The test piece is of a standard shape and size, its central cylinder being subjected to torsion. A jaw, carrying the handle and a drum on which paper is wrapped, takes a firm hold of one end of the test-piece, whose further end lies in another jaw rigidly connected with a heavy pendulum carrying a pencil. By a continuous slow motion of the handle the pendulum is gradually deviated more and more from the vertical, through the intervention of the test-piece. The axis of the test-piece lies in the axis of motion. This motion of the pendulum by means of a curved guide causes an axial (parallel to axis of



Relaxing the strain at E, we find that { BC = angular recovery; OB = " " set.

Fig. 218 § 222

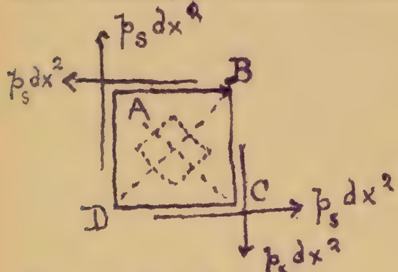


Fig. 219. § 223.

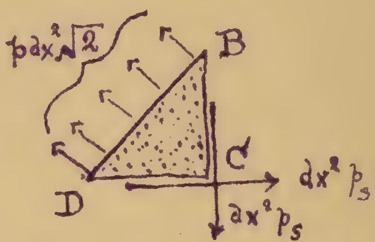


Fig. 220. § 223

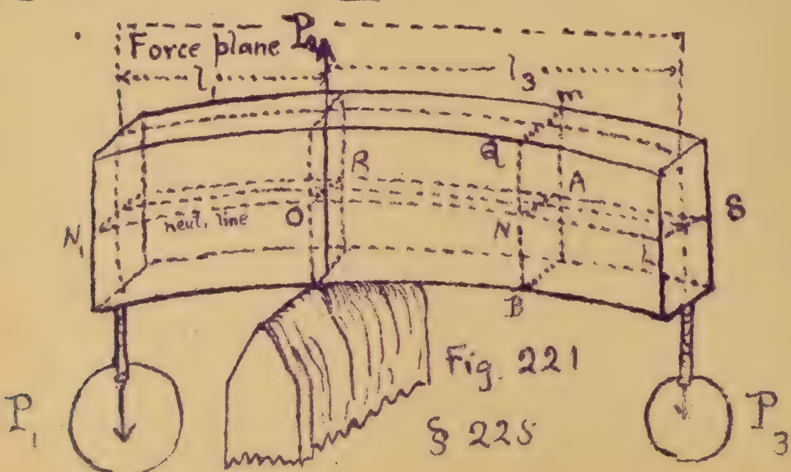


Fig. 221 § 225

test-piece) motion of the pencil, as well as an angular deviation equal to that of the pendulum, and this axial distance of the pencil from its initial position measures the moment of torsion. As the piece twists, the drum and paper move under the pencil through an angle equal to the angle of torsion so far attained. The abscissa and ordinate of the curve thus ~~marked on the paper~~, measure, when the paper is unrolled, the values of α and Pa through all the stages of the torsion. Fig. 218 shows typical curves thus obtained. Many valuable indications are given by these strain diagrams as to homogeneousness of composition, ductility, etc., etc. On relaxing the strain at any stage within the elastic limit, the pencil retraces its path; but if beyond that limit a new path is taken called an "elasticity-line", in general parallel to the first part of the line, and showing the amount of angular recovery and the permanent angular set.

223. EXAMPLES IN TORSION. The modulus of safe shearing strength, S', as given in § 221, is expressed in pounds per square inch; hence these two units should be adopted throughout in any numerical examples where one of the above values for S' is used. The same statement applies to the modulus of shearing elasticity, E_s, in the Table § 210.

EXAMPLE 1. Fig. 216. With $P = 1$ ton, $a = 3$ ft. $l = 10$ ft., and the radius of the cylindrical shaft $r = 2.5$ inches, required the max. shearing stress per sq. inch, p_s, the shaft being of wro't iron. From eq. (3) § 216

$$p_s = \frac{Pae}{I_p} = \frac{2000 \times 36 \times 2.5}{\frac{1}{2}\pi} = 2930. \text{ lbs per}$$

sq. inch which is a safe value for any ferrous metal.

EXAMPLE 2. What H.P. is the shaft in Ex. 1 transmitting, if it makes 50 revolutions per minute? Let u = numb. of revolutions per unit of time and N = the number of units of work per unit of time constituting one horse-power. Then $H.P. = Pu\,2\pi a \div N$, which for the foot-pound-minute system of units gives

$$H.P. = 2000 \times 50 \times 2\pi \times 3 \div 33000 = 57\tfrac{1}{4}\ H.P.$$

Example 3. What different radius should be given to the shaft in Ex. 1. if two radii at its extremities, originally parallel are to make an angle of 2° when the given moment of torsion is acting, the strains in the shaft remaining constant. From eq. (4) § 217, and the table § 210, with $\alpha = \frac{2°}{180°}\pi = 0.035$ (π-measure), and $I_p = \frac{1}{2}\pi r^4$, we have

$$r^4 = \frac{2000 \times 36 \times 120}{\frac{1}{2}\pi\, 0.035 \times 9000000} = 17.45 \therefore r = 2.04 \text{ inches.}$$

(This would bring about a different p_s but still safe.) The foregoing is an example in stiffness.

Example 4. A working shaft of steel (solid) is to transmit 4000 H.P. and make 60 rev. per minute, the maximum twisting moment being $1\frac{1}{2}$ times the average; required its diameter.

Example 5. In example 1. $p_s = 2930$ lbs. per square inch; what tensile stress does this imply on a plane at 45° with the pair of planes on which p_s acts? Fig. 219 shows a small cube, of edge $= dx$, (taken from the outer helix of fig. 215,) free and in equilibrium, the plane of the paper being tangent to the cylinder; while 220 shows the portion BDC, also free, with the unknown total tensile stress $p\,dx^2\sqrt{2}$ acting on the newly exposed rectangle of area $= dx \times dx\sqrt{2}$, p being the unknown stress per unit of area. From symmetry the stress on this diagonal plane has no shearing component. Putting Σ [components normal to BD] $= 0$ we have

$$p\,dx^2\sqrt{2} = 2\,dx^2 p_s \cos 45° = dx^2 p_s \sqrt{2} \quad \therefore p = p_s \quad (1)$$

That is, a normal tensile stress exists in the diagonal plane of the cubical element equal in intensity to the shearing stress on one of the faces, i.e., = 2930 lbs. per sq. in. in this case.

Similarly in the plane AC will be found a compressive stress of 2930 lbs. per sq. in. If a plane surface had been exposed making any other angle than 45° with the face of the cube in fig. 219, we should have found shearing and normal stresses each less than p_s per sq. inch. Hence the interior dotted cube in 219, if shown "free" (fig. 221), is in tension in one direction in compression in the other and with no shear, these normal stresses having equal intensities. Since S' is usually less than T' or C', if p_s is made = S' the tensile and compressive actions are not injurious. It follows therefore that when a cylinder is in torsion any helix at an angle of 45° with the axis is a line of tensile, or of compressive stress, according as it is a right or left handed "corkscrew", or vice versâ.

Example 6. A solid and a hollow cylindrical shaft, of equal length, contain the same amount of the same kind of metal, the solid one fitting the hollow of the other. Compare their torsional strengths, used separately.

CHAP. III.

Flexure of Homogeneous Prisms under perpendicular forces in one plane.

224. ASSUMPTIONS OF THE COMMON THEORY OF FLEXURE. This theory is sufficiently exact for ordinary engineering purposes if the constants employed are properly determined by a wide range of experiments, and involves certain assumptions of as simple a nature as possible, consistently with practical facts. These assumptions

are as follows, (for prisms and for solids with variable cross sections when the cross sections are similarly situated as regards a central straight axis) viz.:

(1) The external or "applied" forces are all perpendicular to the axis of the piece and lie in one plane, which may be called the force-plane; the force-plane contains the axis of the piece and cuts each cross-section symmetrically;

(2) The cross-sections remain plane surfaces during flexure.

(3) There is a surface which is parallel to the axis and perpendicular to the force-plane and along which the elements of the solid experience no tension nor compression in an axial direction, this being called the NEUTRAL SURFACE;

(4) The projection of the neutral surface upon the force plane being called the NEUTRAL LINE or ELASTIC CURVE, the bending or flexure of the piece is so slight that an elementary division, ds, of the neutral line may be put $= dx$, its projection on a line parallel to the direction of the axis before flexure;

(5) The elements of the body contained between any two consecutive cross sections, whose intersections with the neutral surface are the respective NEUTRAL AXES of the sections, experience elongations (or contractions, according as they are situated on one side or the other of the neutral surface), in an axial direction, whose amounts are proportional to their distances from the neutral axis and indicate corresponding tensile or compressive stresses;

(6) $E_t = E_c$;

(7) The dimensions of the cross section are small compared with the length of the piece;

(8) There is no shear perpendicular to the force plane on internal surfaces perpendicular to that plane.

Fig. 222 to 226 To face p. 42. §§ 226 ··· 233

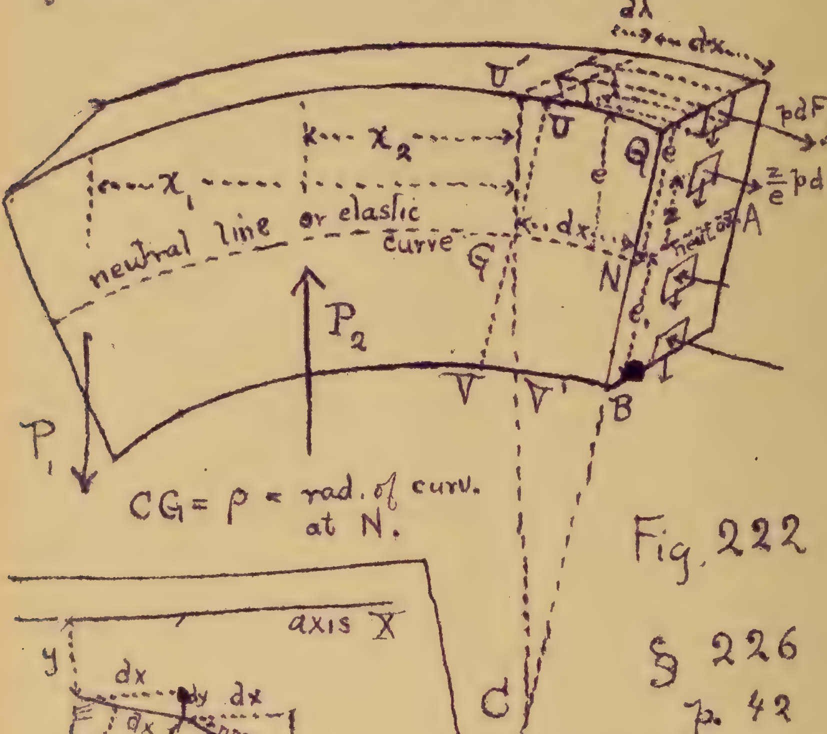


Fig. 222

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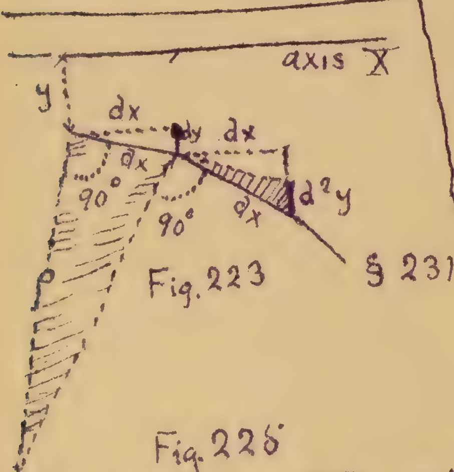


Fig. 223 § 231

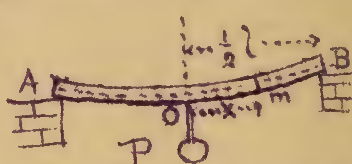
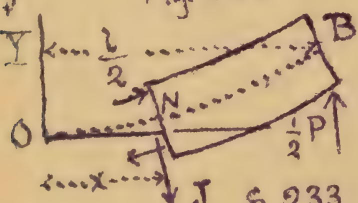


Fig. 224 § 233

Fig. 225



§ 233.

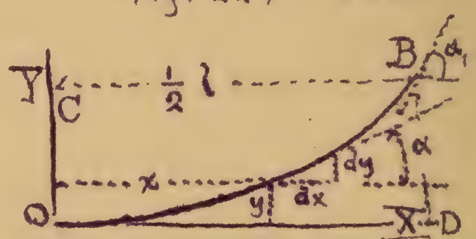


Fig. 226 § 233

In the locality where any one of the external forces is applied, local stresses are of course induced which demand separate treatment. These are not considered at present.

225. ILLUSTRATION. Consider the case of flexure shown in Fig. 221. The external forces are three, (neglecting the weight of the beam) viz.: P_1, P_2, and P_3. P_1 and P_3 are loads, P_2 the reaction of the support.

The force plane is vertical. NL is the neutral line or elastic curve. NA is the neutral axis of the cross-section at m; this cross-section, originally perpendicular to the sides of the prism, is during flexure ⊥ to their tangent planes drawn at the intersection lines; in other words, the side view QNB_1 of any cross-section is perpendicular to the neutral line. In considering the whole prism free we have the system P_1, P_2, and P_3 in equilibrium, whence from $\Sigma Y = 0$ we have $P_2 = P_1 + P_3$, and from Σ(mom about O) $= 0$, $P_3 l_3 = P_1 l_1$. Hence given P_1 we may determine the other two external forces. A reaction such as P_2 is sometimes called a supporting force. The elements above the neutral surface N_1OLS are in tension; those below in compression (in an axial direction)

226. THE ELASTIC FORCES. Conceive the beam in Fig. 221 separated into two parts by any transverse section such as QA, and the portion NON_1 considered as a free body in fig. 222. Of this free body the surface QAB is one of the bounding surfaces; but was originally an internal surface of the beam in Fig. 221. Hence in Fig. 222 we must put in the stresses acting on all the dF's or elements of area of QAB. These stresses represent the actions of the body taken away upon the body which is left, and according to assumptions (5), (6) and (8) consist of normal stresses (tension or compression) proportional per unit of area, to the distance, z, of the dF's from the neutral axis, and of shearing stresses parallel to the force-

plane, (which in most cases will be vertical).

The intensity of this shearing stress on any dF varies with the position of the dF with respect to the neutral axis, but the law of its variation will be investigated later (§§ 253 and 254). These stresses, called the ELASTIC FORCES of the cross section exposed, and the external forces P_1 and P_2, form a system in equilibrium. We may therefore apply any of the conditions of equilibrium proved in § 38.

227. THE NEUTRAL AXIS CONTAINS THE CENTRE OF GRAVITY OF THE CROSS-SECTION. Fig. 222. Let e = the distance of the outermost element of the cross-section from the neutral axis and the normal stress per unit of area upon it be $= p$ whether tension or compression. Then by assumptions (5) and (6), § 224, the intensity of normal stress on any dF is $= \frac{z}{e}p$ and the actual normal stress on any dF is $= \frac{z}{e}p\,dF$ - - - - - - - - (1)

This equation is true for dF's having negative z's, i.e. on the other side of the neutral axis, the negative value of the force indicating normal stress of the opposite character; for if the relative elongation (or contraction) of two axial fibres is the same for equal z's, one above, the other below, the neutral surface, the stresses producing the elongations are also the same, provided $E_t = E_c$; see §§ 184 and 201.

For this free body in equilibrium put $\Sigma X = 0$, (X is a horizontal axis). Put the normal stresses equal to their X components, the flexure being so slight, and the X component of the shears $= 0$ for the same reason. This gives (see eq. (1))

$$\int \frac{z}{e} p\,dF = 0;\ \text{i.e.}\ \frac{p}{e}\int dF z = 0;\ \text{or,}\ \frac{p}{e}F\bar{z} = 0 \quad \text{-- -- (2)}$$

In which $\bar{z}$ = distance of the centre of gravity of the cross-section from the neutral axis, from which, though unknown in position, the z's have been measured; (see eq. (4) § 23).

In eq. (2) neither $p \div e$ nor F can be zero $\therefore \bar{z}$ must $= 0$;

i.e. the neutral axis contains the centre of gravity. Q.E.D. [If the external forces were not all perpendicular to the beam this result would not be obtained, necessarily]

228. THE SHEAR. The "total shear", or simply the "shear", in the cross-section is the sum of the vertical shearing stresses on the respective dF's. Call this sum J, and we shall have from the free body in fig. 222, by putting $\Sigma Y = 0$ (Y being vertical)

$$P_2 - P_1 - J = 0 \therefore J = P_2 - P_1 \quad ----- \quad (3)$$

That is, the shear equals the algebraic sum of the external forces acting on one side (only) of the section considered. This result implies nothing concerning its mode of distribution over the section.

229. THE MOMENT. By the "Moment of Flexure", or simply the Moment, at any cross-section is meant the sum of the moments of the elastic forces of the section, taking the neutral axis as an axis of moments. In this summation the normal stresses appear alone, the shear taking no part, having no lever arm about the neutral axis. Hence fig. 222 the moment of flexure $= \int\left(\frac{z}{e}\,p\,dF\right)z = \frac{p}{e}\int dF z^2 = \frac{pI}{e}$ ---- (4)

This function, $\int dF z^2$, of the cross-section or plane figure is the quantity called MOMENT OF INERTIA in § 85. For the free body in fig. 222, by putting Σ (mom.s about the neutral axis NA) $= 0$, we have then

$$\frac{pI}{e} - P_1 x_1 + P_2 x_2 = 0, \text{ or, in general, } \frac{pI}{e} = M \quad -- \quad (5)$$

in which M signifies the sum of the moments, about the neutral axis of the section, of all the forces acting on the free body considered, exclusive of the elastic forces of the exposed section itself.

230. STRENGTH IN FLEXURE. Eq. (5) is available for solving problems involving the STRENGTH of beams and girders, since it contains p, the greatest normal stress per unit of area to be found in the section.

In the cases of the present chapter, where all the external forces are perpendicular to the prism or beam, and have therefore no components parallel to the beam, i.e. to the axis X, it is evident that the normal stresses in any section, as QB fig. 222, are equivalent to a couple; for the condition $\Sigma X = 0$ falls entirely upon them and cannot be true unless the resultant of the tensions is equal, parallel, and opposite to that of the compressions; These two equal and parallel resultants, not being in the same line, form a couple (§28), which we may call the stress-couple. The moment of this couple is the moment of flexure $\frac{pI}{e}$, and it is further evident that the remaining forces in fig. 222, viz.: the shear J and the external forces P_1 and P_2 are equivalent to a couple of equal and opposite moment to the one formed by the normal stresses.

231. FLEXURAL STIFFNESS. The neutral line, or elastic curve, containing the centres of gravity of all the sections, was originally straight; its radius of curvature at any point, as N. fig. 222, during flexure may be introduced as follows. QB and U'V' are two consecutive cross-sections, originally parallel, but now inclined so that the intersection O found by prolonging them sufficiently is the centre of curvature of the ds (put $= dx$) which separates them at N, and $CG = \rho$ = the radius of curvature of the elastic curve at N. From the similar triangles U'UG and GNC we have $d\lambda : dx :: e : \rho$, in which $d\lambda$ is the elongation, U'U, of a portion, originally $= dx$, of the outer fibre. But the relative elongation $\varepsilon = \frac{d\lambda}{dx}$ of the latter is, by §184, within the elastic limit, $= \frac{p}{E}$ $\therefore \frac{p}{E} = \frac{e}{\rho}$ and eq. (5) becomes

$$\frac{EI}{\rho} = M \quad - - - - \quad (6)$$

From (6) the radius of curvature can be computed. E = the value of $E_t = E_c$, as ascertained from experiments in bending.

To obtain a differential equation of the elastic curve (6) may be transformed thus, fig. 223. The curve being very flat, consider two consecutive ds's with equal dx's; they may be put = their dx's. Produce the first to intersect the dy of the second, thus cutting off the d^2y, i.e. the difference between two consecutive dy's. Drawing a perpendicular to each ds at its left extremity, the centre of curvature C is determined by their intersection, and thus the radius of curvature ρ. The two shaded triangles have an equal angle $d\phi$, and d^2y is nearly perpendicular to the prolonged dx; hence, considering them similar, we have $\rho : dx :: dx : d^2y$ ∴

$\frac{1}{\rho} = \frac{d^2y}{dx^2}$ and hence from eq. (6) we have

$$(approx.) \quad \text{- - - -} \; EI\frac{d^2y}{dx^2} = M \quad \text{- - - -} \qquad (7)$$

as a differential equation of the elastic curve. From this the equation of the elastic curve may be found, the deflections at different points computed, and an idea thus formed of the stiffness. All beams in the present chapter being prismatic and homogeneous both E and I are the same (i.e. constant) at all points of the elastic curve. In using (7) the axis X must be taken parallel to the length of the beam before flexure, which must be slight.

232. RESILIENCE OF FLEXURE. If the external forces are made to increase gradually from zero up to certain maximum values some of them may do work, by reason of their points of application moving through certain distances due to the yielding, or flexure, of the body. If at the beginning and also at the end of this operation the body is at rest, this work has been expended on the elastic resistance of the body and an equal amount, called the work of resilience (or springing-back), will be restored by the elasticity of the body if released from the external forces, provided the elastic limit has not been passed. The energy thus temporarily stored is of the potential kind; see §§ 148, 180, 196 and 218.

ELASTIC CURVES.

233. CASE I. HORIZONTAL PRISMATIC BEAM, SUPPORTED AT BOTH ENDS, WITH A CENTRAL LOAD. WEIGHT OF BEAM NEGLECTED. Fig. 224.

First considering the whole beam free, we find each reaction to be $=\frac{1}{2}P$. AOB is the neutral line; required the equation of the portion OB referred to O as an origin and to the tangent line through O as the axis of X. To do this consider as free the portion mB between any section m on the right of O and the near support, in fig. 225. The forces holding this free body in equilibrium are the one external force $\frac{1}{2}P$, and the elastic forces acting on the exposed surface. The latter consist of J, the shear, and the tensions and compressions represented in the figure by their equivalent "stress-couple". Selecting N, the neutral axis of m, as an axis of moments (that J may not appear in the moment equation) and putting $\Sigma(\text{mom})=0$ we have

$$\frac{P}{2}\left(\frac{l}{2}-x\right)-EI\frac{d^2y}{dx^2}=0 \;\therefore\; EI\frac{d^2y}{dx^2}=\frac{P}{2}\left(\frac{l}{2}-x\right) \quad \text{(1)}$$

Fig. 226 shows the elastic curve OB in its purely geometrical aspect, much exaggerated.

Eq. (1) gives the second x-derivative of y equal to a function of x. Hence the first x-derivative of y will be equal to the x-anti-derivative of that function, plus a constant, C. (By anti-derivative is meant the converse of derivative, sometimes called integral though not in the sense of summation). Hence from (1) we have (EI being a constant factor remains undisturbed)

$$EI\frac{dy}{dx}=\frac{P}{2}\left(\frac{l}{2}x-\frac{x^2}{2}\right)+C \qquad (2)'$$

(2)′ is an equation between two variables $dy \div dx$ and x, and holds good for any point between O and B; $dy \div dx$ denoting the tang. of α, the <u>slope</u> or angle between the tangent-line and X. At O the slope is zero, and x also zero; hence at O (2)′ becomes

$$EI \times 0 = 0 - 0 + C$$

which enables us to determine the constant C, whose value must be the same at O as for all points of the curve. Hence $C=0$ and (2)′ becomes

$$EI\frac{dy}{dx}=\frac{P}{2}\left(\frac{l}{2}x-\frac{x^2}{2}\right) \quad \text{-----} \quad (2)$$

from which the slope, tan α, (or simply α, in π-measure; since the angle is small) may be found at any point. Thus at B we have $x=\frac{l}{2}$ and $dy \div dx = \alpha_1$, and

$$\therefore \alpha_1=\frac{1}{16}\cdot\frac{Pl^2}{EI}$$

Again, taking the x-anti-derivative of both members of eq. (2) we have

$$EIy=\frac{P}{2}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right)+C' \qquad (3)'$$

and since at O both x and y are zero, C' is zero. Hence the equation of the elastic curve OB is

$$EIy=\frac{P}{2}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right) \quad \text{--} \quad (3)$$

To compute the deflection of O from the right line joining A and B in fig. 224, i.e. BD, $=d$, we put $x=\frac{l}{2}$ in (3), y being then $=d$, and obtain

$$BD=d=\frac{1}{48}\cdot\frac{Pl^3}{EI} \qquad (4)$$

Eq. (3) does not admit of negative values for x; for if the free body of fig. 225 extended to the left of O, the external forces acting would be P, downward, at O; and $\frac{P}{2}$, upward, at B, instead of the latter alone; thus altering the form of eq. (1). From symmetry, however, we know that the curve AO fig 224 is symmetrical with OB about the vertical through O.

233.a. LOAD SUDDENLY APPLIED. Eq. (4) gives the deflection d corresponding to the force or pressure P applied at the middle of the beam and is seen to be proportional to it. If a load G hangs at rest from the middle of the beam, $P=G$; but if the load G, being initially placed at rest upon the unbent beam, is suddenly released from the external constraint necessary to hold it there, it sinks and deflects the beam, the pres-

Figs. 227 to 234 To face p. 49 §§ 234 to 241

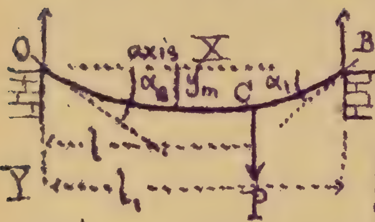


Fig 227 § 234

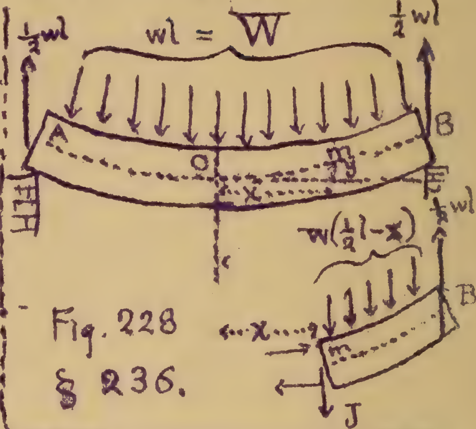


Fig. 228 § 236.

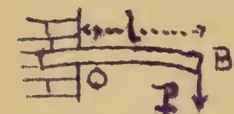


Fig. 229 § 237

Fig. 230

Fig. 231 § 238

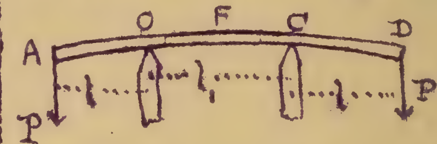
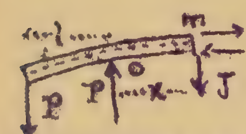


Fig. 232 § 238



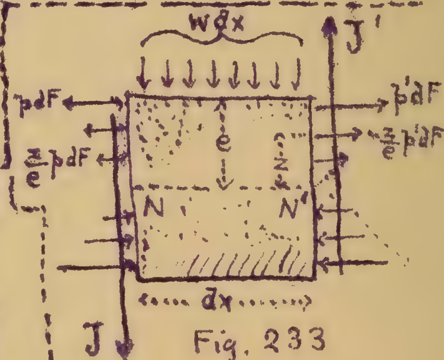


Fig. 233 § 240

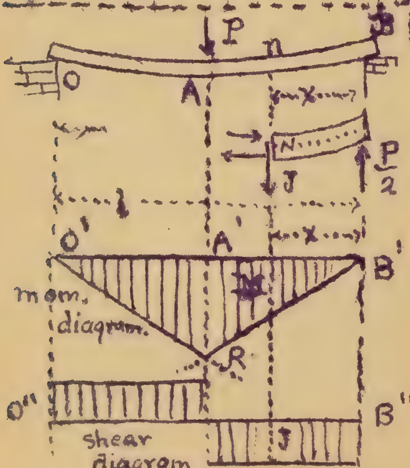


Fig. 234 § 241

sure P actually felt by the beam varying with the deflections as the load sinks. What is the ultimate deflection d_m? Let P_m = the pressure between the load and the beam at the instant of maximum deflection. The work so far done in bending the beam $= \frac{1}{2} P_m d_m$. The potential energy given up by the load $= G d_m$, while its initial and final kinetic energies are both nothing.

$$\therefore G d_m = \frac{1}{2} P_m d_m \qquad (5)$$

That is $P_m = 2G$. Since at this instant the load is subjected to an upward force of $2G$ and to a downward force of only G (gravity) it immediately begins an upward motion reaching the point whence the motion began, and thus the oscillation continues: We here suppose the elasticity of the beam unimpaired. This is called the "sudden" application of a load, and produces, as shown above, double the pressure on the beam which it does when gradually applied, and a double deflection. The work done by the beam in raising the weight again is called its resilience.

Similarly, if the weight G is allowed to fall on the middle of the beam from a height h, we shall have

$$G \times (h + d_m), \text{ or approx., } Gh, = \frac{1}{2} P_m d_m ;$$

and hence, since (4) gives d_m in terms of P_m,

$$Gh = \frac{1}{96} \cdot \frac{P_m^2 l^3}{EI} ; \text{ or } Gh = \frac{24\, EI d_m^2}{l^3} \qquad (6)$$

This theory supposes the mass of the beam small compared with the falling weight.

234. CASE II. HORIZONTAL PRISMATIC BEAM, SUPPORTED AT BOTH ENDS, BEARING A SINGLE ECCENTRIC LOAD. WEIGHT OF BEAM NEGLECTED. Fig. 227. The reactions of the points of support, P_0 and P_1, are easily found by considering the whole beam free, and putting first $\Sigma(\text{mom.})_0 = 0$, whence $P_1 = Pl_1 \div l$, and then $\Sigma(\text{mom.})_B = 0$, whence $P_0 = Pl_1 \div (l - l_1)$. P_0 and P_1 will now be treated as known quantities.

The elastic curves OC and CB, though having a common

tangent line at C (and hence the same slope α_c), and a common ordinate at C, have separate equations and are both referred to the same origin and axes, as shown in the figure. The slope at O, α_0, and that at B, α_1, are unknown constants, to be determined in the progress of the work.

EQUATION OF OC. Considering as free a portion of the beam extending from B to a section made anywhere on OC, x and y being the co-ordinates of the neutral axis of that section, we conceive the elastic forces put in on the exposed surface, as in the preceding problem, and put Σ(mom. about neutral axis of the section) = 0 which gives

$$EI\frac{d^2y}{dx^2} = P(l-x) - P_1(l_1-x)\,; \qquad (1)$$

whence, by taking the x anti-derivatives of both members

$$EI\frac{dy}{dx} = P\left(lx-\frac{x^2}{2}\right) - P_1\left(l_1x-\frac{x^2}{2}\right) + C$$

To find C, write out this equation for the point O, where $dy \div dx = \alpha_0$ and $x = 0$, and we have $C = EI\alpha_0$; hence the equation for slope is

$$EI\frac{dy}{dx} = P\left(lx-\frac{x^2}{2}\right) - P_1\left(l_1x-\frac{x^2}{2}\right) + EI\alpha_0 \quad (2)$$

Again taking the x anti-derivatives, we have from (2)

$$EIy = P\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) - P_1\left(\frac{l_1x^2}{2}-\frac{x^3}{6}\right) + (C'=0) \quad (3)$$

(at O both x and y are = 0 ∴ $C'=0$) In equations (1), (2), and (3) no value of x is to be used <0 or $>l_1$, since for points in CB different relations apply, thus:

EQUATION OF CB. Fig. 227. Let the free body extend from B to a section made anywhere on CB. Σ(moms) as before = 0, gives

$$EI\frac{d^2y}{dx^2} = -P_1(l_1-x) \qquad (4)$$

(N.B. In (4), as in (1), $EI\,d^2y \div dx^2$ is written equal to a negative quantity because itself essentially negative; for

the curve is concave to the axis X in the first quadrant of the co-ordinate axes).

From (4) we have in the ordinary way (x anti-deriv.)

$$EI\frac{dy}{dx} = -P_1\left(l_1x - \frac{x^2}{2}\right) + C'' \dots \quad (5)'$$

To determine C'', consider that the curves CB and OC have the same slope $(dy \div dx)$ at C where $x = l$; hence put $x = l$ in the right-hand members of (2) and of (5)′ and equate the results. This gives $C'' = \frac{1}{2}Pl^2 + EI\alpha_0$ and ∴

$$EI\frac{dy}{dx} = \frac{Pl^2}{2} + EI\alpha_0 - P_1\left[l_1x - \frac{x^2}{2}\right] \dots\dots (5)$$

and ∴

$$EIy = \frac{Pl^2}{2}x + EI\alpha_0 x - P_1\left[l_1\frac{x^2}{2} - \frac{x^3}{6}\right] + C''' \dots (6)'$$

At C, where $x = l$, both curves have the same ordinates; hence, by putting $x = l$ in the right hand members of (3) and (6)′ and equating results, we obtain $C''' = -\frac{1}{6}Pl^3$. ∴ (6)′ becomes

$$EIy = \frac{1}{2}Pl^2x + EI\alpha_0 x - P_1\left[\frac{l_1x^2}{2} - \frac{x^3}{6}\right] - \frac{Pl^3}{6} \dots\dots (6)$$

as the **EQUATION OF CB**, Fig. 227. But α_0 is still an unknown constant, to find which write out (6) for the point B where $x = l_1$ and $y = 0$, whence we obtain

$$\alpha_0 = \frac{1}{6EIl_1}\left[Pl^3 - 3Pl^2l_1 + 2P_1l_1^3\right] \dots\dots (7)$$

α_1 = a similar form, putting P_0 for P_1, and $(l_1 - l)$ for l.

235. MAXIMUM DEFLECTION in CASE II; of Fig. 227. The ordinate y_m of the lowest point is thus found: Assuming $l > \frac{1}{2}l_1$, it will occur in the curve OC. Hence put the $dy \div dx$ of that curve, as expressed in equation (2), = 0. Also for α_0 write its value from (7), having put $P_1 = Pl \div l_1$, and we have

$$P\left(lx - \frac{x^2}{2}\right) - P\frac{l}{l_1}\left(l_1x - \frac{x^2}{2}\right) + \frac{1}{6}\frac{Pl}{l_1}\left(l^2 - 3ll_1 + 2l_1^2\right) = 0$$

whence $\left[x \text{ for max. } y\right] = \sqrt{\frac{1}{3} l (2l_1 - l)}$ Now substitute this value of x in (6), also α_0 from (7), and put $P_1 = Pl \div l_1$, whence

$$\text{MAX. DEFLEC.} = y_{max} = \frac{1}{9} \cdot \frac{P}{EIl_1}\left[l^3 - 3l^2 l_1 + 2l l_1^2\right]\sqrt{\frac{1}{3} l (2l_1 - l)}$$

236. CASE III. HORIZONTAL PRISMATIC BEAM SUPPORTED AT BOTH ENDS AND BEARING A UNIFORMLY DISTRIBUTED LOAD ALONG ITS WHOLE LENGTH. (The weight of the beam itself, if considered, constitutes a load of this nature.) Let l = the length of the beam and w = the weight, per unit of length, of the loading; then the load coming upon any length x will be $= wx$, and the whole load $= wl$. By hypothesis w is constant. Fig. 228. From symmetry we know that the reactions of A and B are each $= \frac{1}{2} wl$, that the middle O of the neutral line is its lowest point, and the tangent line at O is horizontal. Conceiving a section made at any point of the neutral line at a distance x from O, consider as free the portion of beam on the right of m. The forces holding this portion in equilibrium are $\frac{1}{2} wl$, the reaction at B; the elastic forces of the exposed surface at m, viz.: the tensions and compressions, forming a couple, and J the total shear; and a portion of the load, $W(\frac{1}{2} l - x)$. The sum of the moments of these latter forces about the neutral axis of m, is the same as that of their resultant, (i.e. their sum since they are parallel), and this resultant acts in the middle of the length $\frac{1}{2} l - x$. Hence the sum of these moments $= W(\frac{1}{2} l - x)\frac{1}{2}(\frac{1}{2} l - x)$. Now putting Σ (mom. about neutral axis of m) $= 0$ for this free body, we have

$$EI\frac{d^2 y}{dx^2} = \frac{1}{2} Wl(\tfrac{1}{2} l - x) - \frac{1}{2} W(\tfrac{1}{2} l - x)^2;$$

i.e. $$EI\frac{d^2 y}{dx^2} = \frac{1}{2} W(\tfrac{1}{4} l^2 - x^2) \qquad (1)$$

Taking the x anti-derivative of both sides of (1),

$$EI\frac{dy}{dx}=\frac{1}{2}W\left(\frac{1}{4}l^2x-\frac{1}{3}x^3\right)+(C=0) \qquad (2)$$

as the equation of slope. (The constant is $=0$ since at O both $dy \div dx$ and x are $=0$.) From (2),

$$EIy=\frac{1}{2}W\left(\frac{1}{8}l^2x^2-\frac{1}{12}x^4\right)+[C'=0] \quad - - - \quad (3)$$

which is the equation of the elastic curve; throughout, i.e., it admits any value of x from $x=+\frac{1}{2}l$ to $x=-\frac{1}{2}l$. This is an equation of the fourth degree, one degree higher than those for the Curves of Cases I and II, where there were no distributed loads. If w were not constant, but proportional to the ordinates of an inclined right line, eq. (3) would be of the fifth degree; if w were proportional to the vertical ordinates of a parabola with axis vertical, (3) would be of the sixth degree; and so on.

By putting $x=\frac{1}{2}l$ in (3) we have the deflection of O below the horizontal through A and B, viz.: (with W = total load = wl)

$$d=\frac{5}{384}\cdot\frac{wl^4}{EI}=\frac{5}{384}\cdot\frac{Wl^3}{EI} \qquad (4)$$

237. CASE IV. CANTILEVERS. A horizontal beam whose only support consists in one end being built in a wall, as in fig. 229; or supported as in fig. 230; is sometimes called a cantilever. Let the student prove that in fig. 229, with a single end load P, the deflection of B below the tangent at O is $d=\frac{1}{3}Pl^3 \div EI$; the same statement applies to fig. 230, but the tangent at O is not horizontal if the beam was originally so. The greatest deflection of the elastic curve from the right line joining AB, in fig. 230 is evidently given by the equation for x max. in §235, by writing, instead of P of that equation, the reaction at O in fig. 230. This assumes that the max. deflection occurs between A and O. If it occurs between O and B put (l_1-l) for l.

If in fig. 229 the loading is uniformly distributed along the beam at the rate of w pounds per linear unit, the student may

also prove that the deflection of B below the tangent at O is

$$d = \frac{1}{8} wl^4 \div EI = \frac{1}{8}\frac{Wl^3}{EI}$$

238. CASE V. HORIZONTAL PRISMATIC BEAM BEARING EQUAL TERMINAL LOADS AND SUPPORTED SYMMETRICALLY AT TWO POINTS.

Fig. 231. Weight of beam neglected. In the preceding cases we have made use of the approximate form $EI d^2y \div dx^2$ in determining the forms of elastic curves. In the present case the elastic curve from O to C is more directly dealt with by employing the more exact expression $EI \div \rho$ (see §231) for the moment of the stress-couple in any section. The reactions at O and C are each $= P$, from symmetry. Considering free a portion of the beam extending from A to any section m between O and C (Fig. 232) we have, by putting Σ (mom. about neutral axis of m) $= 0$,

$$P(l+x) - \frac{EI}{\rho} - Px = 0 \quad \therefore \rho = \frac{Pl}{EI}$$

That is, the radius of curvature is the same at all points of OC; in other words OC is the arc of a circle with the above radius. The upward deflection of F from the right line joining O and C can easily be computed from a knowledge of this fact. This is left to the student as also the value of the slope of the tangent line at O (and C). The deflection of D from the tangent at $C = \frac{1}{3} Pl^3 \div EI$, as in fig. 229.

SAFE LOADS IN FLEXURE.

239. MAXIMUM MOMENT. As we examine the different sections of a given beam under a given loading we find different values of p, the normal stress per unit of area in the outer element, as obtained from eq. (5) §229, viz.:

$$\frac{pI}{e} = M. \qquad (1)$$

in which I is the "Moment of Inertia" (§85) of the plane figure formed by the section, about its neutral axis, e the distance of the most distant (or outer) fibre from the neutral

axis, and M the sum of the moments, about this neutral axis, of all the forces acting on the free body of which the section in question is one end, exclusive of the stresses on the exposed surface of that section. In other words M is the sum of the moments of the forces which balance the stresses of the section, these moments being taken about the neutral axis of the section under examination.

For the prismatic beams of this chapter e and I are the same at all sections, hence p varies with M and becomes a maximum when M is a maximum. In any given case the location of the "dangerous section", or section of maximum M, and the amount of that maximum value may be determined by inspection and trial, this being the only method if the external forces are detached. If, however, the loading is continuous according to a definite algebraic law the calculus may often be applied, taking care to treat separately each portion of the beam between two consecutive reactions of supports or detached loads.

As a graphical representation of the values of M along the beam in any given case, these values may be conceived as laid off as vertical ordinates (according to some definite scale) from a horizontal axis just below the beam. If the upper fibres are in compression in any portion of the beam, so that that portion is convex downwards, these ordinates will be laid off below the axis, and vice versâ; for it is evident that at a section where $M=0$, p also $=0$, i.e. the character of the normal stress in the outermost fibre changes (from tension to compression, or vice versâ) when M changes sign. It is also evident from eq. (6) §231 that the radius of curvature changes sign, and consequently the curvature is reversed, when M changes sign. These moment ordinates form a MOMENT DIAGRAM, and their extremities a MOMENT CURVE.

The maximum moment, M_m, being found, in terms of the loads and reactions, we must make the p of the "dangerous section", where $M=M_m$, equal to a safe value R', and thus may write

$$\frac{R'I}{e} = M_m \qquad (2)$$

Eq. (2) is available for finding any one unknown quantity, whether it be a load, span, or some one dimension of the beam, and is concerned only with the STRENGTH, and not with the stiffness of the beam. If it is satisfied in any given case, the normal stress on all elements in all sections is known to be = or < R', and the design is therefore safe in that one respect

As to danger arising from the shearing stresses in any section, the consideration of the latter will be taken up in a subsequent chapter and will be found to be necessary only in beams composed of a thin web uniting two flanges. The total shear, however, denoted by J, bears to the moment M, an important relation of great service in determining M_m. This relation, therefore, is presented in the next article.

240. THE SHEAR IS THE FIRST x-DERIVATIVE OF THE MOMENT. Fig. 233 (x is the distance of any section, measured parallel to the beam, from an arbitrary origin) Consider as free a vertical slice of the beam included between any two consecutive vertical sections whose distance apart is dx. The forces acting are the elastic forces of the two internal surfaces now laid bare, and, possibly, a portion, $w dx$, of the loading, which at this part of the beam has some intensity = w lbs. per running linear unit. Putting Σ (mom. about axis N') = 0 we have

$$\frac{pI}{e} - \frac{p'I}{e} + J\,dx + w\,dx \cdot \frac{dx}{2} = 0$$

But $\frac{pI}{e} = M$, the Moment of the left hand section, $\frac{p'I}{e} = M'$, that of the right; whence we may write, after dividing through by dx and transposing,

$$\frac{M'-M}{dx} = J + w\frac{dx}{2} \quad \text{i.e.} \quad \frac{dM}{dx} = J; \qquad (3)$$

for $w\frac{dx}{2}$ vanishes when added to the finite J. This proves the theorem.

Now the value of x which renders M a maximum or minimum

would be obtained by putting the derivative $dM \div dx =$ zero; hence we may state as a

COROLLARY. At sections where the moment is a maximum or minimum the shear is zero.

The shear J at any section is easily determined by considering free the portion of beam from the section to either end of the beam and putting Σ (vertical components) $= 0$.

In this article the words maximum and minimum are used in the same sense as in calculus; i.e. graphically, they are the ordinates of the moment curve at points where the tangent line is horizontal. If the moment curve reduced to a straight line, or a series of straight lines, it has no maximum or minimum in the strict sense just stated; nevertheless the relation is still practically borne out by the fact that at the sections of greatest and least ordinates in the moment diagram the shear changes sign suddenly. This is best shown by drawing a shear diagram, whose ordinates are laid off vertically from a horizontal axis and under the respective sections of the beam. They will be laid off upward or downward according as J is found to be upward or downward, when the free body considered extends from the section toward the right.

In these diagrams the moment ordinates are set off on an arbitrary scale of so many inch-pounds, or foot-pounds, to the linear inch of paper; the shears, being simply pounds, or some other unit of force, on a scale of so many pounds to the inch of paper. The scale on which the beam is drawn is so many feet, or inches, to the inch of paper.

241. SAFE LOAD AT THE MIDDLE OF A PRISMATIC BEAM SUPPORTED AT THE ENDS. Fig. 234. The reaction at each support is $\frac{1}{2}P$. Make a section n at any distance $x < \frac{l}{2}$ from B. Consider the portion nB free, putting in the proper elastic and external forces. The weight of beam is neglected. From Σ (moms. about n) $= 0$ we have

$$\frac{pI}{e} = \frac{P}{2}x\text{; i.e. } M = \frac{1}{2}Px$$

Evidently M is proportional to x, and the ordinates representing it will therefore be limited by the straight line B'R, forming a triangle B'RA'. From symmetry, another triangle O'RA' forms the other half of the moment diagram. From inspection, the maximum M is seen to be in the middle where $x = \frac{1}{2}l$, and hence

$$(M\ max) = M_m = \frac{1}{4}Pl \quad - - - - - \quad (1)$$

Again by putting Σ (vert. compons.) = 0, for the free body nB we have

$$J = \frac{P}{2}$$

and must point downwards since $\frac{P}{2}$ points upwards. Hence the shear is constant and $= \frac{1}{2}P$ at any section in the right hand half. If n be taken in the left half we would have, nB being free, from Σ (vert. com.) = 0,

$$J = P - \frac{1}{2}P = \frac{1}{2}P$$

the same numerical value as before; but J must point upward, since $\frac{P}{2}$ at B and J at n must balance the downward P at A. At A, then, the shear changes sign suddenly, that is, passes through the value zero; also at A M is a maximum, thus illustrating the statement in §240. Notice the shear diagram in fig. 234.

To find the safe load in this case we write the maximum value of the normal stress, $p_1 = R'$, a safe value; (see table in a subsequent article) and solve the equation for P. But the maximum value of p is in the outer fibre at A, since M for that section is a maximum. Hence

$$\frac{R'I}{e} = \frac{1}{4}Pl \qquad (2)$$

is the equation for safe loading in this case, so far as the normal stresses in any section are concerned.

242. SAFE LOAD UNIFORMLY DISTRIBUTED ALONG A PRISMATIC BEAM SUPPORTED AT THE ENDS. Let the load per lineal unit of the length of beam be = w; (this can be made to include the weight of

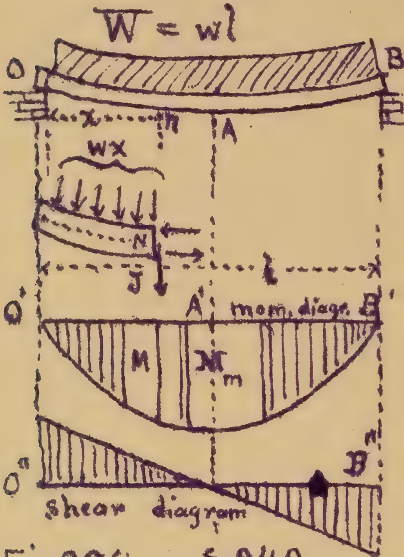


Fig. 235 § 242

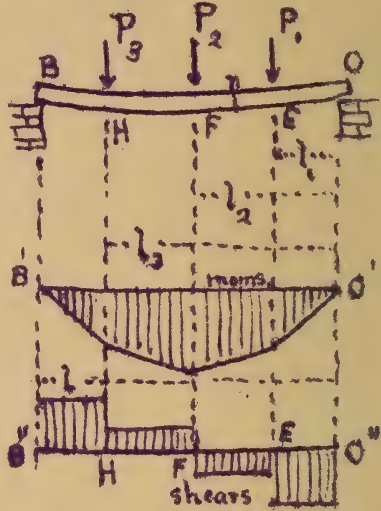


Fig. 236 § 242

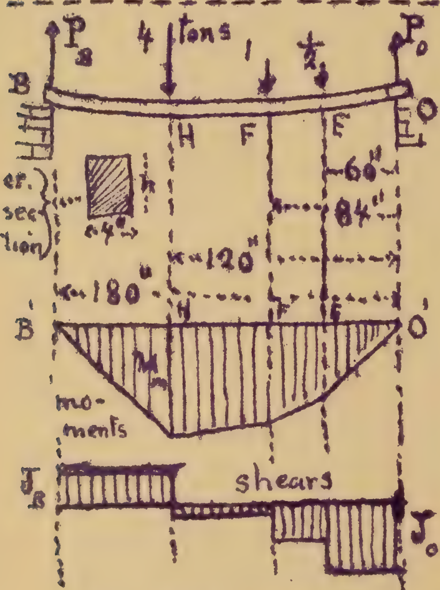


Fig. 237 § 244

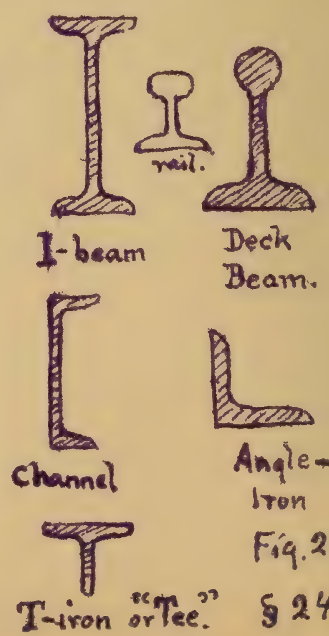


Fig. 238

§ 248

of the beam itself). Fig. 235. From symmetry, each reaction $= \frac{1}{2} wl$. For the free body nB we have, putting Σ (mom. about n) $= 0$,

$$\frac{pI}{e} = \frac{wl}{2}x - (wx)\frac{x}{2} \therefore M = \frac{w}{2}(lx - x^2)$$

which gives M for any section by making x vary from 0 to l. Notice that in this case the law of loading is continuous along the whole length, and that hence the moment curve is continuous for the whole length.

To find the shear J, at n, we may either put Σ (vert. compons.) $= 0$ for the free body, whence $J = \frac{1}{2}wl - wx$ and must therefore be downward for a small value of x; or, employing §240, we may write $dM \div dx = 0$, which gives

$$J = \frac{dM}{dx} = \frac{w}{2}(l - 2x) \qquad (1)$$

the same as before. To find the max. M, or M_m, put $J = 0$, which gives $x = \frac{1}{2}l$. This indicates a maximum, for when substituted in $d^2M \div dx^2$, i.e. in $-wx$, a negative result is obtained. Hence M_m occurs at the middle of the beam and its value is

$$M_m = \frac{1}{8}wl^3; \therefore \frac{R'I}{e} = \frac{1}{8}wl^3 = \frac{1}{8}Wl^2 \qquad (2)$$

is the equation of safe loading.

It can easily be shown that the moment curve is a portion of a parabola, whose vertex is at A'' under the middle of the beam, and axis is vertical. The shear diagram consists of ordinates to a simple straight line inclined to its axis and crossing it, i.e. giving a zero shear, under the middle of the beam, where we find the max. M.

If a frictionless dovetail joint with vertical faces were introduced at any locality in the beam and thus divided the beam into two parts, the presence of J would be made manifest by the downward slipping of the left hand part on the right hand part if the joint were on the right of the middle, and vice versâ if it were on the left of the middle. This shows why the

ordinates in the two halves of the shear diagrams have opposite signs. The greatest shear is close to either support and is $J_m = \frac{1}{2} wl$.

243. PRISMATIC BEAM SUPPORTED AT ITS EXTREMITIES AND LOADED IN ANY MANNER. EQUATION FOR SAFE LOADING. Fig. 236. Given the loads P_1, P_2, and P_3, whose distances from the right support are l_1, l_2, and l_3; required the equation for safe loading; i.e., find M_m and write it $= R'I \div e$.

If the moment curve were continuous, i.e., if M were a continuous function of x from end to end of the beam, we could easily find M_m by making $dM \div dx = 0$, i.e., $J = 0$, and substitute the resulting value of x in the expression for M. But in the present case of detached loads, J is not zero necessarily, at any section of the beam. Still there is some one section where it changes sign, i.e., passes suddenly through the value zero, and this will be the section of greatest moment (though not a maximum in the strict sense used in Calculus). By considering any portion nO as free, J is found equal to THE REACTION AT O DIMINISHED BY THE LOADS OCCURRING BETWEEN n and O. The reaction at O is

$$P_0 = (P_1 l_1 + P_2 l_2 + P_3 l_3) \div l$$

obtained by treating the whole beam as free (in which case no elastic forces come into play) and putting Σ(mom. about B) $= 0$.

If n is taken anywhere between O and E, $J = P_0$
" " " " " " E " F, $J = P_0 - P_1$
" " " " " " F " H, $J = P_0 - P_1 - P_2$
" " " " " " H " B, $J = P_0 - P_1 - P_2 - P_3$

This last value of J also = the reaction at the other support B. Accordingly, the shear diagram is seen to consist of a number of horizontal steps. The relation $J = dM \div dx$ is such that the <u>slope</u> of the moment curve is proportional to the <u>ordinate</u> of the shear diagram, and that for a sudden change in the <u>slope</u> of the mo-

ment curve there is a sudden change in the shear ordinate. Hence in the present instance, I being constant between any two consecutive loads, the moment curve reduces to a straight line between the same loads, this line having a different inclination under each of the portions into which the beam is divided by the loads. Under each load the slope of the moment curve and the ordinate of the shear diagram change suddenly. In fig. 236 the shear passes through the value zero, i.e., changes sign, at F; or algebraically we are supposed to find that $P_0 - P_1$ is + while $P_0 - P_1 - P_2$ is −, in the present case. Considering FO, then, as free, we find M_m to be

$$M_m = P_0 l_2 - P_1(l_2 - l_1)$$ and the equation for safe loading is

$$\frac{R'I}{e} = P_0 l_2 - P_1(l_2 - l_1) \qquad (1)$$

(if the max. M is at F.) It is also evident that the greatest shear is equal to the reaction at one or the other support, whichever is the greater; and that the moment at either support is zero.

The student should not confuse the moment curve, which is entirely imaginary, with the neutral line (or elastic curve) of the beam itself. The greatest moment is not necessarily at the section of maximum deflection of the neutral line.

For the case in fig. 236 we may therefore state that the max. moment, and consequently the greatest tension or compression in the outer fibre, will be found in the section under that load for which the sum of the loads (including this load itself) between it and either support first equals or exceeds the reaction of that support. The amount of this moment is then obtained by treating as free either of the two portions of the beam into which this section divides the beam.

244. NUMERICAL EXAMPLE OF THE PRECEDING ARTICLE. Fig. 237. Given P_1, P_2, P_3, equal to $\frac{1}{2}$ ton, 1 ton, and 4 tons, respectively; $l_1 = 5$ feet, $l_2 = 7$ feet, and $l_3 = 10$ feet; while the total length is 15 feet. The beam is of timber, of rectangular cross-section, the horizontal width being $b = 10$ inches, and the value of R' (greatest safe normal stress) $= \frac{1}{2}$ ton per sq. inch, or 1000 lbs. per sq. inch.

Required the proper depth h for the beam, for safe loading.

SOLUTION. Adopting a definite system of units, viz. the inch-ton-second system, we must reduce all distances such as l_1, etc. to inches, express all forces in tons, write $R' = \frac{1}{2}$ (tons per sq. inch) and interpret all results by the same system. Moments will be in inch-tons, and shears in tons. [N.B. In problems involving the strength of materials the inch is more convenient as a linear unit than the foot, since any stress expressed in lbs, or tons, per sq. inch is numerically 144 times as small as if referred to the square foot.]

Making the whole beam free, we have from moms. about O

$$P_0 = \frac{1}{180}\left[\frac{1}{2}\times 60 + 1\times 84 + 4\times 120\right] = 3.3 \text{ tons}$$

The shear anywhere between O and E is $J = +3.3$ tons.

" " " " " E and F is $J = 3.3 - \frac{1}{2} = 2.8$ tons.

" " " " " F and H is $J = 3.3 - \frac{1}{2} - 1 = +1.8$ tons.

" " " " " H and B $= 3.3 - \frac{1}{2} - 1 - 4 = -2.2$ tons.

Hence since on passing H the shear changes sign, the max. moment is at H, and making HO free has a value

$$M_m = 3.3\times 120 - \frac{1}{2}\,60 - 1\times 36 = 320 \text{ inch tons}$$

For safety M_m must $= \frac{R'I}{e}$, in which $R' = \frac{1}{2}$ ton per sq. inch, $e = \frac{1}{2}$ of h the unknown depth of the beam, and I, from § 90, $= \frac{1}{12}bh^3$, with $b = 10$ inches. These substitutions give

$$\frac{1}{12}\cdot\frac{1}{2}\cdot\frac{2}{h}\times 10h^3 = 320; \text{ or } h^2 = 384 \text{ sq. inches } \therefore h = 19.6 \text{ in.}$$

245. COMPARATIVE STRENGTH OF RECTANGULAR BEAMS. For such a beam, under a given loading, the equation for safe loading is

$$\frac{R'I}{e} = M_m \quad \text{i.e. } \frac{1}{6}R'bh^2 = M_m;$$

whence the following is evident, (since for the same length, mode of support, and distribution of load, M_m is proportional to the safe loading). For rectangular prismatic beams of the same length, same material, same mode of support and same arrangement of load;

(1) The safe load is proportional to the width of beams having

the same depth (h).

(2) The safe load is proportional to the square of the depth of beams having the same width (b).

(3) The safe load is proportion to the depth of beams having the same volume (i.e. the same bh)

(It is understood that the sides of the section are horizontal and vertical respectively and that the material is homogeneous.)

246. COMPARATIVE STIFFNESS OF RECTANGULAR BEAMS. Taking the deflection under the same loading as an inverse measure of the stiffness, and noting that in §§ 233, 235, and 236, this deflection is inversely proportional to $I = \frac{1}{12}bh^3$ = the "moment of inertia" of the section about its neutral axis, we may state that:

For rectangular prismatic beams of the same length, same material, same mode of support, and same loading;

(1) The stiffness is proportional to the width for beams of the same depth.

(2) The stiffness is proportional to the cube of the height for beams of the same width (b).

(3) The stiffness is proportional to the square of the depth for beams of equal volume (bhl)

(4) If the length alone vary, the stiffness is inversely proportional to the cube of the length.

247. TABLE OF MOMENTS OF INERTIA. These are here recapitulated for the simpler cases, and also the values of e, the distance of the outermost fibre from the axis.

Since the stiffness varies as I (other things being equal), while the strength varies as $I \div e$, it is evident that a square beam has the same stiffness in any position (§ 89), while its strength is greatest with one side horizontal, for then e is smallest being $= \frac{1}{2}b$.

Since for any cross-section $I = \int dF z^2$, in which z = the distance of any element, dF, of area from the neutral axis, a beam is made both stiffer and stronger by throwing most of its material into two flanges united by a vertical web, thus forming a so called

"I-beam" of a I shape. But not without limit, for the web must be thick enough to cause the flanges to act together as a solid of continuous substance, and, if too high, is liable to buckle sideways, thus requiring lateral stiffening. These points will be treated later.

Section	I	e
Rectangle, width = b, depth = h	$\frac{1}{12} b h^3$	$\frac{1}{2} h$
Hollow rectangle, symmet. about neutral axis (— b_2 —, h_2; h_1, b_1)	$\frac{1}{12}[b_1 h_1^3 - b_2 h_2^3]$	$\frac{1}{2} h_1$
Triangle, width = b, depth = h	$\frac{1}{36} b h^3$	$\frac{2}{3} h$
Circle of radius r	$\frac{1}{4} \pi r^4$	r
Ring of concentric circles (r_1, r_2)	$\frac{1}{4} \pi (r_1^4 - r_2^4)$	r_1
Rhombus (b, h)	$\frac{1}{48} b h^3$	$\frac{1}{2} h$
Square with side b vertical	$\frac{1}{12} b^4$	$\frac{1}{2} b$
" " " " b at 45°	$\frac{1}{12} b^4$	$\frac{1}{2}\sqrt{2}\, b$

248. MOMENT OF INERTIA OF I-BEAMS, BOX-GIRDERS, etc. In common with other large companies, the N.J. Steel and Iron Co. of Trenton, N.J. (Cooper, Hewett, and Co.) manufacture prismatic rolled beams of wrought-iron variously called I-beams, deck-beams, rails, and shape iron, (including channels, angles, tees, etc.) according to the form of section. See fig. 238 for these forms. The company publishes a pocket-book giving tables of quantities relating to the strength and stiffness of beams, such as the safe loads for various spans, moments of inertia of their sections in various positions, etc., etc. The moments of inertia of I-beams and deck-beams are computed according to §§ 92 and 93, with the inch as linear unit. The I beams range from 4 in. to about 15 inches deep, the deck-beams being about 7 and 8 in. deep.

For beams of still greater stiffness and strength combinations of plates, channels, angles, etc. are riveted together, forming "built-beams". The proper design for the riveting of such beams will

be examined later. For the present the parts are assumed to act together as a continuous mass. For example, Fig. 240 shows a "box-girder", formed of two "channels" and two plates riveted together. If the axis of symmetry, N, is to be horizontal it becomes the neutral axis. Let C = the moment of inertia of one channel (as given in the pocket-book mentioned) about the axis N perpendicular to the web of the channel. Then the total *moment of inertia of the combination* is (nearly)

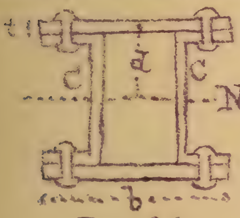


Fig. 240

$$I_N = 2C + 2btd^2 - 4d't'(d-t)^2 \quad \ldots\ldots (1)$$

In (1), b, t, and d are the distances given in Fig. 240 (d extends to the middle of plate) while d′ and t′ are the length and width of a rivet, the former from head to head; (i.e., d′ and t′ are the dimensions of a rivet-hole.)

For example: a box-girder of wrought iron is formed of two 15-inch channels and two plates 10 inches wide and 1 inch thick, the rivet holes ¾ in. wide and 1¾ in. long. That is, $b = 10$; $t = 1$; $d = 8$; $t' = \frac{3}{4}$; and $d' = 1\frac{3}{4}$ inches. Also from the pocket-book we find that for the channel in question, C = 376 biquadratic inches. Hence, eq. (1),

$$I_N = 752 + 2\times 10\times 1\times 64 - 4\times\frac{7}{4}\times\frac{1}{4}(8-1)^2$$

$$= 1946 \text{ biquadr. inches.}$$

Also, since in this instance $e = 8\frac{1}{2}$ inches, and 12000 lbs per sq. inch (or 6 tons per sq. in.) is the value for R′ (= greatest safe normal stress on the outer element of any cross-section) used by the Trenton Co., we have

$$\frac{R'I}{e} = \frac{12000 \times 1946}{8.5} = 2\,747\,280. \text{ inch-lbs.}$$

That is, the box-girder can safely bear a maximum moment

Figs. 240 to 247 To face p. 66 §§ 249 to 253

Fig. 240 will be found on p. 65, in the text.

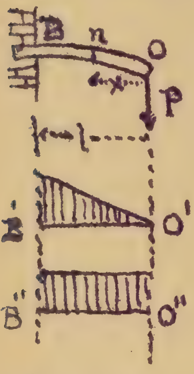


Fig. 241

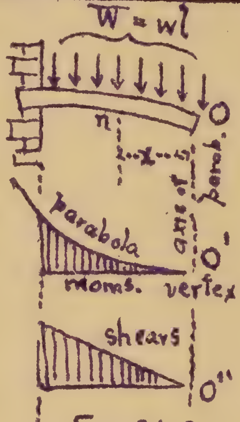


Fig. 242

§ 249.

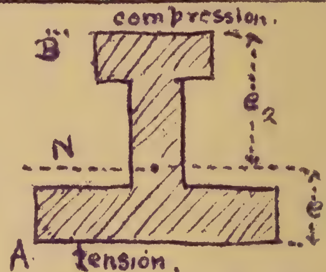


Fig. 243 § 251

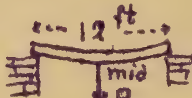


Fig. 244. § 252

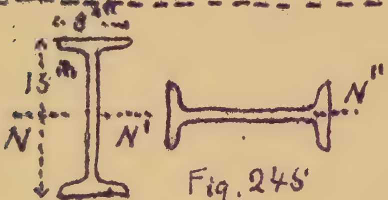


Fig. 245 § 252

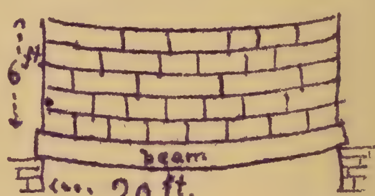


Fig. 245 a § 252

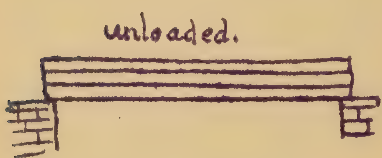


Fig. 246

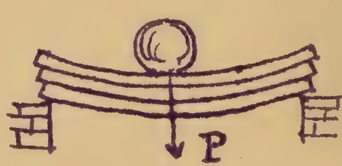


Fig. 247.

§ 253

$M_m = 2\,747\,280$ inch-lbs. $= 1373.6$ inch-tons, as far as the normal stresses in any section are concerned. (Proper provision for the shearing stresses in the section, and in the rivets, will be considered later).

249. STRENGTH OF CANTILEVERS. In fig. 241 with a single concentrated load P at the projecting extremity, we easily find the moment at n to be $M = Px$ and the max. moment to occur at the section next the wall, its value being $M_m = Pl$.

The shear, J, is constant, and $= P$ at all sections. The moment and shear diagrams are drawn in accordance with these results.

If the load $W = wl$ is uniformly distributed on the cantilever, as in fig. 242, by making nO free we have, putting Σ(mom. about n) $= 0$,

$$\frac{pI}{e} = wx\cdot\frac{x}{2} \;\therefore\; M = \tfrac{1}{2}wx^2 \;\therefore\; M_m = \tfrac{1}{2}wl^2 = \tfrac{1}{2}Wl.$$

Hence the moment curve is a parabola, whose vertex is at O' and axis vertical. Putting Σ(vert. compons.) $= 0$ we obtain $J = wx$. Hence the shear diagram is a triangle, and the max. $J = wl = W$.

250. RÉSUMÉ OF THE FOUR SIMPLE CASES. The following table shows the values of the deflections under an arbitrary load P, or W, (within elastic limit), and of the safe load; also the

	Cantilevers		Beams with two end supports	
	with one end load P Fig. 241	with unif. load $W = wl$ Fig. 242	Load P in middle Fig. 234	Unif. load $W = wl$ Fig. 235
Deflection	$\frac{1}{3}\cdot\frac{Pl^3}{EI}$	$\frac{1}{8}\cdot\frac{Wl^3}{EI}$	$\frac{1}{48}\cdot\frac{Pl^3}{EI}$	$\frac{5}{384}\cdot\frac{Wl^3}{EI}$
Safe load (from $\frac{R'I}{e} = M_m$)	$\frac{R'I}{le}$	$2\frac{R'I}{le}$	$4\frac{R'I}{le}$	$8\frac{R'I}{le}$
Relative Strength	1	2	4	8
Relative stiffness under same load	1	$\frac{8}{3}$	16	$\frac{128}{5}$
Relative stiffness under safe load	1	$\frac{4}{3}$	4	$\frac{16}{5}$
Max. Shear $= J_m$ (and location)	P, (at wall)	W, (at wall)	$\frac{1}{2}P$, (at supp.)	$\frac{1}{2}W$, (at supp.)

relative strength, the relative stiffness (under the same load), and the relative stiffness under the safe load, for the same beam.

The max. shear will be used to determine the proper web-thickness for I beams and "built girders". The student should carefully study the foregoing table, noting especially the relative strength, stiffness, and stiffness under safe load, of the same beam.

Thus, a beam with two end supports will bear a double load, if uniformly distributed instead of concentrated in the middle, but will deflect $\frac{1}{5}$ more; whereas with a given load uniformly distributed the deflection would be only $\frac{5}{8}$ of that caused by the same load in the middle, provided the elastic limit is not surpassed in either case.

251. R′, etc. FOR VARIOUS MATERIALS. The formula $\frac{p_m I}{e} = M_m$, from which in any given case of flexure we can compute the value of p_m the greatest normal stress in any outer element, provided all the other quantities are known, holds good theoretically within the elastic limits only. Still, some experimenters have used this formula for the rupture of beams by flexure, calling the value of p_m thus obtained the Modulus of Rupture, R. R is found to differ considerably from both the T or C of §203 with some materials being generally intermediate in value between them. This might be expected, since even supposing the relative extension or compression (i.e. strain) of the fibres to be proportional to their distances from the neutral axis as the load increases toward rupture, the corresponding stresses not being proportional to these strains beyond the elastic limit, no longer vary directly as the distances from the neutral axis.

The following table gives average values for R, R′, R″, and E for the ordinary materials of construction. E, the modulus of elasticity for use in the formulae for deflection, is given as computed from experiments in flexure, and is nearly the same as E_t and E_c.

In any example involving R′, e is usually written equal to the distance of the outer fibre from the neutral axis whether that fibre is to be in tension or compression; since in most materials not only is the tensile equal to the compressive stress for a given strain (relative extension or contraction) but the elastic limit is reached at a-

out the same strain both in tension and compression.

Table for use in Examples in Flexure.

	Timber	Cast Iron	Wro't Iron	Steel
Max. safe stress in outer fibre $= R'$ (lbs. per sq. inch)	1000	6000 in tens. 12000 in comp.	12 000	15 000 to 40 000
Stress in outer fibre at Elastic limit $= R'$ (lbs. per sq. inch			17 000	70 000
"Modul. of Rupture" $= R =$ lbs. per sq. inch	4000 to 20000	40000	60 000	120 000
$E =$ Mod. of Elasticity $=$ lbs. per sq. inch	1000 000 to 3000 000	17 000 000	25 000 000	

In the case of cast iron, however, (see § 203) the elastic limit is reached in tension with a stress $= 9000$ lbs. per sq. inch and a relative extension of $\frac{66}{1000}$ of one per cent., while in compression the stress must be about double to reach the elastic limit, the relative change of form (strain) being also double. Hence with cast iron beams, once largely used but now almost entirely displaced by rolled wrought-iron beams, an economy of material was effected by making the outer fibre on the compressed side twice as far from the neutral axis as that on the stretched side. Thus, fig. 243, cross-sections with unequal flanges were used, so proportioned that the centre of gravity was twice as near to the outer fibre in tension as to that in compression, i.e., $e_2 = 2e_1$; in other words more material is placed in tension than in compression. The fibre A being in tension (within elastic limit) that at B, since it is twice as far from the neutral axis and on the other side, is contracted twice as much as A is extended; i.e. is under a compressive strain double the tension at A, but in accordance with the above figures its state of strain is proportionally as much within the elastic limit as that of A.

Steel beams are gradually coming into use, and may ultimately

replace those of wrought iron.

The great range of values of R for timber is due not only to the fact that the various kinds of wood differ widely in strength, while the behavior of specimens of any one kind depends somewhat on age, seasoning, etc., but also to the circumstance that the size of the beam under experiment has much to do with the result. The experiments of Prof. Lanza at the Mass. Institute of Technology in 1881 were made on full size lumber (spruce), of dimensions such as are usually taken for floor beams in buildings, and gave much smaller values of R (from 3200 to 8700 lbs. per sq. inch) than had previously been obtained. The loading employed was in most cases a concentrated load midway between the two supports.

These low values are probably due to the fact that in large specimens of ordinary lumber the continuity of its substance is more or less broken by cracks, knots, etc., the higher values of most other experimenters having been obtained with small, straight-grained, selected pieces, from one foot to six feet in length.

The value $R' = 12000$ lbs. per sq. inch is employed by the N.J. Iron and Steel Co. in computing the safe loads for their rolled wrought iron beams, with the stipulation that the beams (which are high and of narrow width) must be secure against yielding sideways. If such is not the case the ratio of the actual safe load to that computed with $R' = 12000$ is taken less and less as the span increases. The lateral security referred to may be furnished by the brick arch-filling of a fire-proof floor, or by light lateral bracing with the other beams.

252. NUMERICAL EXAMPLES.

Example 1. A square bar of wrought iron, $1\frac{1}{2}$ in. in thickness is bent into a circular arc whose radius is 200 ft.; the plane of bending being parallel to the side of the square. Required the greatest normal stress p_m in any outer fibre.

SOLUTION. From §§ 230 and 231 we may write

$$\frac{EI}{\rho} = \frac{pI}{e} \quad \therefore p = eE \div \rho, \text{ i.e. is constant.}$$

For the units inch and pound (viz. those of the table in §251) we have $e = \frac{3}{4}$ in., $\rho = 2400$ in. and $E = 25\,000\,000$ lbs. per sq. inch, and $\therefore$

$p = p_m = \frac{3}{4} \times 25\,000\,000 \div 2400 = 7812$. lbs. per sq. in. which is quite safe. At a distance of $\frac{1}{2}$ inch from the neutral axis, the normal stress is $= \left[\frac{1}{2} \div \frac{3}{4}\right] p_m = \frac{2}{3} p_m = 5208$ lbs. per sq. in. (If the force-plane (i.e. plane of bending) were parallel to the diagonal of the square, e would $= \frac{1}{2} \times 1.5\sqrt{2}$ inches, giving $p_m = [7812 \times \sqrt{2}]$ lbs. per sq. in.) §238 shows an instance where a portion, OC fig. 231, is bent in a circular arc.

EXAMPLE 2. A hollow cylindrical cast-iron pipe of radii $3\frac{1}{2}$ and 4 inches is supported and loaded as in fig. 244. Required the safe load, neglecting the weight of the pipe. From the table in §250 we have for safety

$$P = 4\frac{R'I}{le}$$

From §251 we put $R' = 6000$ lbs. per sq. in; and from §247 $I = \frac{\pi}{4}(r_1^4 - r_2^4)$; and with these values r_2 being $= \frac{7}{2}$, $r_1 = 4$, $e = r_1 = 4$, $\pi = \frac{22}{7}$ and $l = 144$ inches (the inch must be the unit of length since $R' = 6000$ lbs. per sq. INCH) we have

$$P = 4 \times 6000 \times \frac{1}{4} \cdot \frac{22}{7}(256 - 150) \div [144 \times 4]$$

$$\therefore P = 3470. \text{ POUNDS.}$$

The weight of the beam itself is $G = V\gamma$, i.e.,

$$G = \pi(r_1^2 - r_2^2) l \gamma = \frac{22}{7}(16 - 12\tfrac{1}{4}) 144 \times \frac{450}{1728} = 443 \text{ lbs.,}$$

(See §7 and notice that γ, here, must be lbs. per cubic inch) and this weight being a uniformly distributed load is equivalent to half as much, 221 lbs., applied in the middle, as far as the strength of the beam is concerned (see §250), $\therefore P$ must be taken $= 3249$ lbs. when the weight of the beam is considered.

EXAMPLE 3. A wrought-iron rolled I-beam supported at the ends is to be loaded uniformly Fig. 235, the span being equal to 20 feet. Its cross section, fig. 245, has a depth

parallel to the web, of 15 inches, a flange width of 5 inches, in the pocket-book of the Trenton Co. it is called a 15 inch light I-beam, weighing 150 lbs per yard, with a moment of inertia = 523 bi-quad. inches about a gravity axis perpendicular to the web (i.e. when the web is vertical, the strongest position) and = 15 biq.-in. about a gravity axis parallel to the web (i.e. when the web is placed horizontally).

First placing the web vertical, we have from § 250, W_1 Safe load, distributed, $= 8\frac{R'I_1}{le_1}$. With $R' = 12000$,

$I_1 = 523$, $l = 240$ inches, $e_1 = 7\frac{1}{2}$ inches, this gives $W_1 = [8 \times 12000 \times 523] \div [240 \times \frac{15}{2}] = 27902$ lbs. But this includes the weight of the beam, $G = 20^{ft.} \times \frac{150^{lbs}}{3} = 1000$ lbs.; hence a distributed load of 26902 lbs. or 13.45 tons may be placed on the beam, (secured against lateral yielding.) (The pocket-book referred to gives 13.27 tons as the safe load, but the depth of beam there used is $15\frac{3}{16}$ inches.)

Secondly, placing the web horizontal,

$$W_2 = 8\frac{R'I_2}{le_2} = [8 \times 12000 \times 15] \div [240 \times \tfrac{5}{2}] = \frac{45}{523} \text{ of } W_1$$

or only about $\frac{1}{12}$ of W_1.

EXAMPLE 4. Required the deflection in the first case of Ex. 3. From § 250 the deflection at middle is

$$d_1 = \frac{5}{384} \cdot \frac{W_1 l^3}{EI_1} = \frac{5}{384} \cdot \frac{8R'I_1}{le_1} \cdot \frac{l^3}{I_1} = \frac{5}{48} \cdot \frac{R'}{E} \cdot \frac{l^2}{e_1}$$

i.e. $d_1 = \frac{5}{48} \cdot \frac{12000}{25\,000\,000} \cdot \frac{(240)^2}{\frac{15}{2}}$; (inch and pound). ∴ $d_1 = 0.384''$

EXAMPLE 5. A rectangular beam of yellow pine, of width $b = 4$ inches, is 20 ft. long, rests on two end supports, and is to carry a load of 1200 lbs. at the middle, required the proper depth h. From § 250

$$P = 4\frac{R'I}{le} = 4\frac{R'}{l}\,\frac{bh^3}{12}\,\frac{1}{\frac{1}{2}h}$$

$\therefore h^2 = 6Pl \div 4R'b$. For variety, use the inch and ton. For this system of units, $P = 0.60$ tons, $R' = 0.50$ tons per sq. in., $l = 240$ inches and $b = 4$ inches.

$$\therefore h^2 = (6 \times 0.6 \times 240) \div (4 \times 0.5 \times 4) = 108 \text{ sq. in} \quad \therefore h = 10.4 \text{ in.}$$

EXAMPLE 6. Suppose the depth in Ex. 5 to be determined by the condition that the deflection shall be $= \frac{1}{500}$ of the span or length. We should then have from § 250

$$d = \frac{1}{500}l = \frac{1}{48}\frac{Pl^3}{EI}$$

Using the inch and ton, with $E = 1\,200\,000$ lbs. per sq. inch, which $= 600$ tons per sq. inch, and $I = \frac{1}{12}bh^3$, we have

$$h^3 = \frac{500 \times 0.60 \times 240 \times 240 \times 12}{48 \times 600 \times 4} = 1800 \quad \therefore h = 12.2 \text{ in.}$$

As this is > 10.4 the load would be safe, as well.

EXAMPLE 7. Required the length of a wro't iron pipe supported at its extremities, its internal radius being $2\frac{1}{4}$ in., the external 2.50 in., that the deflection under its own weight may equal $\frac{1}{100}$ of the length.

EXAMPLE 8. Fig. 245. The wall is 6 feet high and one foot thick, of common brick work (see § 7) and is to be borne by an I-beam in whose outer fibres no greater normal stress than 8000 lbs. per sq. inch is allowable. If a number of I-beams is available, ranging in height from 6 in. to 15 in. (by whole inches), which one shall be chosen in the present instance, if their cross-sections are SIMILAR FIGURES, the moment of inertia of the 15-inch beam being 800 biquad. inches?

SHEARING STRESSES IN FLEXURE.

253. SHEARING STRESSES IN SURFACES PARALLEL TO THE NEUTRAL SURFACE. If a pile of boards (see Fig. 246) is used to support a load, the boards being free to slip on each other, it is noticeable that the ends overlap, al-

Fig. 248 § 253

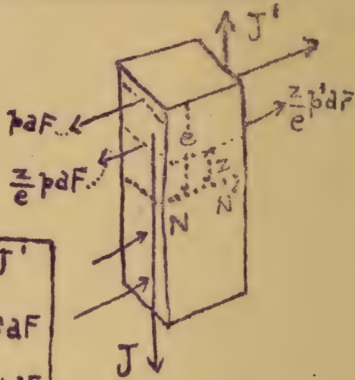


Fig 249 § 253

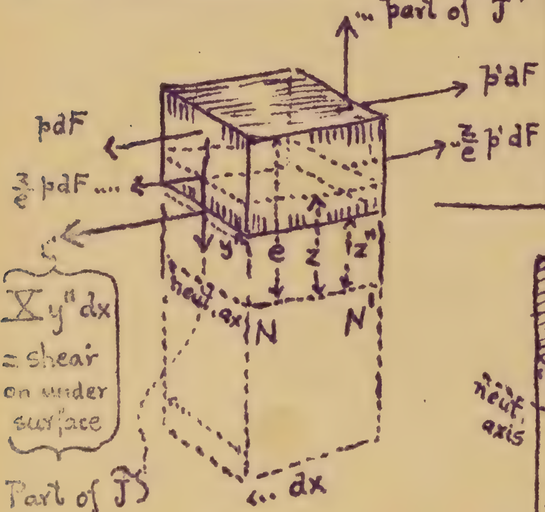


Fig. 250 § 253

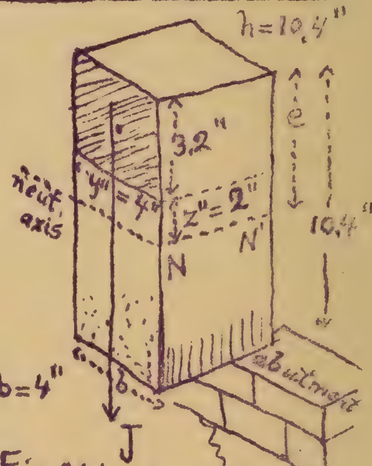
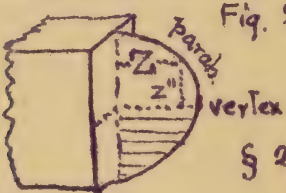


Fig. 251 § 254

Fig. 252



§ 254

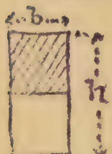


Fig. 253

Fig. 254

Fig. 255

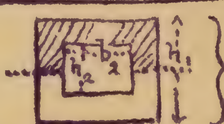


Fig 256

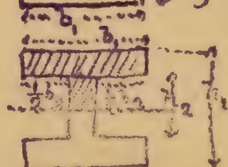


Fig. 257

though the boards are of equal length (now see Fig. 247); i.e. slipping has occurred along the surfaces of contact, the combination being no stronger than the same boards side by side. If, however, they are glued together, piled as in the former figure, the slipping is prevented and the deflection is much less under the same load P. That is, the compound beam is both stronger and stiffer than the pile of loose boards, but the tendency to slip still exists and is known as the "shearing stress in surfaces parallel to the neutral surface." Its intensity per unit of area will now be determined by the usual free-body method. In fig. 248 let AN' be a portion, considered free, on the left of any section N', of a prismatic beam slightly bent under forces in one plane and perpendicular to the beam. The moment equation, about the neutral axis at N', gives

$$\frac{p'I}{e} = M'; \quad \text{whence} \quad p' = \frac{M'e}{I} \quad \ldots\ldots \quad (1)$$

Similarly, with AN as a free body, NN' being $= dx$,

$$\frac{pI}{e} = M; \quad \text{whence} \quad p = \frac{Me}{I} \quad \ldots\ldots \quad (2)$$

p and p' are the respective normal stresses in the outer fibre in the transverse sections N and N' respectively.

Now separate the block NN', lying between these two consecutive sections, as a free body, (in fig. 249.) And furthermore remove a portion of the top of the latter block, the portion lying above a plane passed parallel to the neutral surface and at any distance z'' from that surface. This latter free body is shown in fig. 250, with the system of forces representing the actions upon it of the portions taken away. The under surface, just laid bare, is a portion of a surface (parallel to the neutral surface) in which the above mentioned slipping, or shearing, tendency exists. The lower portion (of the block NN') which is now removed exerted this rubbing, or sliding, force on the remainder along the under surface of the latter. Let the unknown intensity of this shearing force be X (per unit of area);

then the shearing force on this under surface is $= X y'' dx$, (y'' being the horizontal width of the beam at this distance z'' from the neutral axis of N') and takes its place with the other forces of the system, which are the normal stresses between $\int_{z=z''}^{z=e}$, and portions of J and J', the respective total vertical shears. (The manner of distribution of J over the vertical section is as yet unknown; see next article).

Putting Σ (horiz. compons.) $= 0$ in fig. 250, we have

$$\int_{z''}^{e} \frac{z}{e} p' dF - \int_{z''}^{e} \frac{z}{e} p dF - X y'' dx = 0 \quad \therefore X y'' dx = \frac{p'-p}{e} \int_{z''}^{e} z dF$$

But from eqs. (1) and (2), $p'-p = (M'-M)\frac{e}{I} = \frac{e}{I} dM$, while from §240 $dM = J dx$;

$$\therefore X y'' dx = \frac{J dx}{I} \int_{z''}^{e} z dF \quad \therefore X = \frac{J}{I y''} \int_{z''}^{e} z dF \quad \text{-- (3)}$$

as the required intensity <u>per unit of area</u> of the shearing force in a surface parallel to the neutral surface and at a distance z'' from it. It is seen to depend on the "shear" J and the moment of inertia I of the <u>whole</u> vertical section; upon the horizontal thickness y'' of the beam at the surface in question; and upon the integral $\int_{z''}^{e} z dF$, which (from §23) is the <u>product of the area of that part of the vertical section extending from the surface in question to the outer fibre by the distance of the centre of gravity of that part from the neutral surface.</u>

It now follows, from §209, that the intensity (per unit area) of the shear on an elementary area of the <u>vertical cross section</u> of a bent beam, and this intensity we may call Z, is equal to that X, just found, in the horizontal section which is at the same distance (z'') from the neutral axis.

254. MODE OF DISTRIBUTION OF J, THE TOTAL SHEAR, OVER THE VERTICAL CROSS SECTION. The intensity of this shear, Z (lbs. per sq. inch, for instance) has just been proved to be

$$Z = X = \frac{J}{I y''} \int_{z''}^{e} z dF \quad (4)$$

To illustrate this, required the value of Z two inches above the neutral axis, in a cross section close to the abutment, in Ex. 5., § 252. Fig. 251 shows this section. From it we have for the shaded portion, lying above the locality in question, $y'' = 4$ inches, and $\int_{z''=2}^{e=5.2} z\,dF$ = (area of shaded portion) × (distance of its centre of gravity from N) = (12.8 sq. in.) × (3.6 in.) = 46.08 cubic inches.

The total shear J = the abutment reaction = 600 lbs., while $I = \frac{1}{12} bh^3 = \frac{1}{12} \times 4 \times (10.4)^3 = 375$ biquad. inches. Both J and I refer to the whole section.

$$\therefore Z = \frac{600 \times 46.08}{375 \times 4} = 18.42 \text{ lbs. per sq. in.,}$$

quite insignificant. In the neighborhood of the neutral axis, where $z'' = 0$, we have $y'' = 4$ and $\int_{z''=0}^{e} z\,dF = \int_0^e z\,dF = 20.8 \times 2.6 = 54.08$, while J and I of course are the same as before. Hence for $z'' = 0$

$$Z = Z_0 = 21.62 \text{ lbs. per sq. in.}$$

At the outer fibre since $\int_e^e z\,dF = 0$, z'' being $= e$, Z is $= 0$ for a beam of any shape.

For a solid rectangular section like the above, Z and z'' bear the same relation to each other as the coordinates of the parabola in fig. 252.

Since in equation (4) the horizontal thickness, y'', from side to side of the section at the locality where Z is desired, occurs in the denominator, and since $\int_{z''}^{e} z\,dF$ increases as z'' grows numerically smaller, the following may be stated, as to the distribution of J, the shear, in any vertical section, viz.:

The intensity (lbs. per sq. in.) of the shear is zero at the outer elements of the section, and for beams of ordinary shapes is greatest where the section crosses the neutral surface. For forms of cross section having thin webs its value may be so great as

to require special investigation for safe design.

Denoting by Z_0 the value of Z at the neutral axis, (which $= X_0$ in the neutral surface where it crosses the vertical section in question) and putting the thickness of the substance of the beam $= b_0$ at the neutral axis, we have.

$$Z_0 = X_0 = \frac{J}{Ib_0} \times \left\{ \begin{matrix} \text{area above} \\ \text{neutral axis} \\ \text{(or below)} \end{matrix} \right\} \times \left\{ \begin{matrix} \text{the distance of its} \\ \text{cent. grav. from that axis} \end{matrix} \right\} \quad (5)$$

255. VALUES OF Z_0 FOR SPECIAL FORMS OF CROSS SECTION. From the last equation its is plain that for a prismatic beam the value of Z_0 is proportional to J, the total shear, and hence to the ordinate of the shear diagram for any particular case of loading. The utility of such a diagram, as obtained in figs. 234 - 237 inclusive, is therefore evident, for by locating the greatest shearing stress in the beam it enables us to provide proper relations between the loading and the form and material of the beam to secure safety against rupture by shearing.

The table in § 210 gives safe values which the maximum Z_0 in any case should not exceed. It is only in the case of beams with thin webs (see figs. 238 and 240) however, that Z_0 is likely to need attention.

For a RECTANGLE we have, fig. 253, (see eq. 5, § 254) $b_0 = b$, $I = \frac{1}{12} bh^3$, and $\int_0^e z\,dF = \frac{1}{2} bh \cdot \frac{1}{4} h = \frac{1}{8} bh^2$

$$\therefore Z_0 = X_0 = \frac{3}{2} \frac{J}{bh} \quad \text{i.e.,} = \frac{3}{2} (\text{total shear}) \div (\text{whole area})$$

Hence the greatest intensity of shear in the cross-section is $\frac{3}{2}$ as great per unit of area as if the total shear were uniformly distributed over the section.

For a SOLID CIRCULAR section fig. 254

$$Z_0 = \frac{J}{Ib_0} \int_0^e z\,dF = \frac{J}{\frac{1}{4}\pi r^4 . 2r} \cdot \frac{\pi r^2}{2} \cdot \frac{4r}{3\pi} = \frac{4}{3} \cdot \frac{J}{\pi r^2}$$

[See § 26 Prob. 3]

For a HOLLOW CIRCULAR section (concentric circles)

Figs. 258 to 263 To face p. 77 §§ 256 to 260

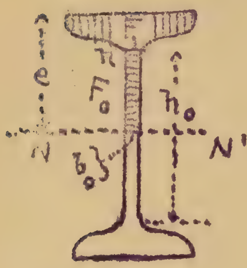


Fig. 258
§ 256

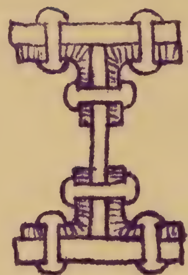


Fig 259
§ 257

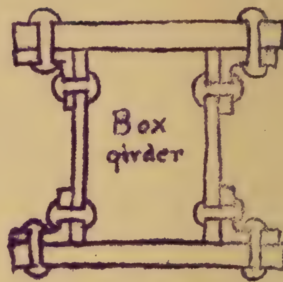


Fig. 260
§ 257

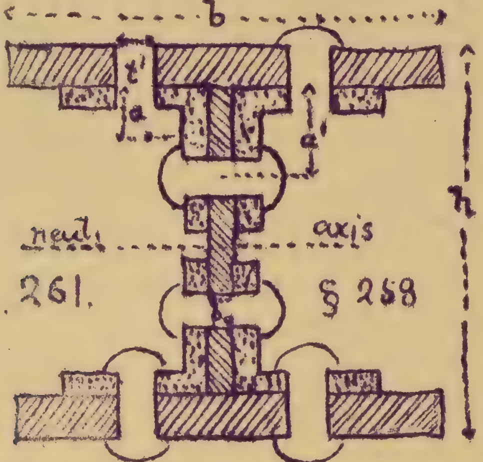


Fig. 261.
§ 258

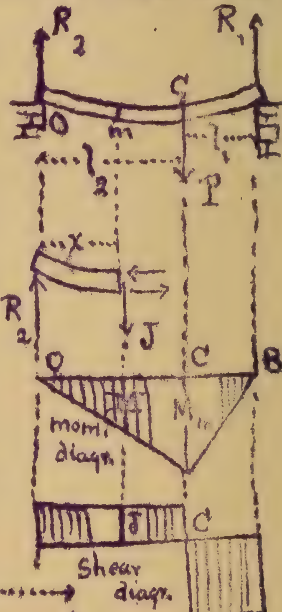


Fig. 262 § 259

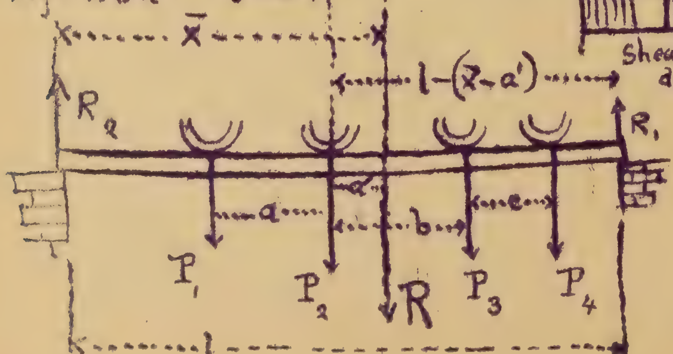


Fig. 263
§ 260

Fig. 255; we have similarly.

$$Z_0 = \frac{J}{\frac{1}{4}\pi(r_1^4 - r_2^4)2(r_1 - r_2)}\left[\frac{\pi r_1^2}{2}\cdot\frac{4r_1}{3\pi} - \frac{\pi r_2^2}{2}\cdot\frac{4r}{3\pi}\right] = \frac{4}{3}\cdot\frac{J(r_1^3 - r_2^3)}{\pi(r_1^4 - r_2^4)(r_1 - r_2)}$$

Applying this formula to Example 2 § 252, we first have as the max. shear $J_m = \frac{1}{2}P = 1735$ lbs., this being the abutment reaction, and hence (putting $\pi = (22 \div 7)$)

$$Z_0 \text{ max.} = \frac{4 \times 7 \times 1735 [64 - 42.8]}{3 \times 22 [256 - 150](4 - 3.5)} = 294. \left\{\begin{array}{l}\text{lbs.}\\ \text{per}\\ \text{sq. in}\end{array}\right.$$

which cast iron is abundantly able to withstand in shearing.

For a HOLLOW RECTANGULAR BEAM, symmetrical about its neutral surface, Fig. 256 (box girder)

$$Z_0 = \frac{J\frac{1}{8}(b_1h_1^2 - b_2h_2^2)}{\frac{1}{12}(b_1h_1^3 - b_2h_2^3)(b_1 - b_2)} = \frac{3}{2}\cdot\frac{J[b_1h_1^2 - b_2h_2^2]}{[b_1h_1^3 - b_2h_2^3][b_1 - b_2]}$$

The same equation holds good for fig. 257 (I-beam with square corners) but then b_2 denotes the sum of the widths of the hollow spaces.

256. SHEARING STRESS IN THE WEB OF AN I-BEAM.

It is usual to consider that, with I-beams (and box beams) with the web vertical the shear J, in any vertical section, is borne exclusively by the web and is uniformly distributed over its section. That this is nearly true may be proved as follows, the flange area being comparatively large. Fig. 258. Let F_1 be the area of one flange, and F_0 that of the half-web. Then since $I = \frac{1}{12}b_0h_0^3 + 2F_1\left(\frac{h_0}{2}\right)^2$, (the last term approximate, being taken as one term of the series $\int z^2 dF$) and $\int_0^e z\,dF = F_1\frac{h_0}{2} + F_0\frac{h_0}{4}$, (the first term approx.) we have

$$Z_0 = \frac{J\int_0^e z\,dF}{I b_0} = \frac{J\frac{1}{4}h_0(2F_1 + F_0)}{\frac{1}{12}h_0^2 b_0(6F_1 + F_0)}, \text{ which} = \frac{J}{b_0h_0}, \text{ if we}$$

neglect F_0 as compared with $2F_1$ and $6F_1$. But b_0h_0 is the area of the whole web, ∴ the shear per unit area at the neutral axis is

nearly the same as if J were uniformly distributed over the web. Similarly, the shearing stress per unit area at n the upper edge of the web is still more nearly equal to $J \div b_0 h_0$ (see eq. 4 §254) for then $\left[\int_{z''=\frac{1}{2}h_0}^{e} z dF\right] = F_1 \frac{1}{2} h_0$ nearly.

The shear per unit area, then, in an ordinary I-beam is obtained by dividing the total shear J by the area of the web section.

EXAMPLE. It is required to determine the proper thickness to be given to the web of the 15-inch wrought-iron rolled beam in Example 3 of §252, the height of web being 13 inches, and a safe shearing stress as low as 4000 lbs. per sq. in. (the practice of the N.J. Steel & Iron Co.); the web being vertical.

The greatest total shear, J_m, occurring at either support and being equal to half the load (see table §250) we have with b_0 = width of web,

$$Z_0 \text{ max.} = \frac{J_m}{b_0 h_0}; \quad \text{i.e. } 4000 = \frac{13950}{b_0 \times 13} \therefore b_0 = 0.26 \text{ inches.}$$

(Units, inch and pound). The 15-inch light beam of the N.J. Co. has a web $\frac{1}{2}$ inch thick, so as to provide for a shear double the value of that in the foregoing example. In the middle of the span $Z_0 = 0$, since $J = 0$.

257. DESIGNING OF RIVETING FOR BUILT BEAMS.

The latter are generally of the I-beam and box forms, made by riveting together a number of continuous shapes, most of the material being thrown into the flange members. E.g. in fig. 257, an I-beam is formed by riveting together, in the manner shown in the figure, a "vertical stem plate" or web, four "angle-irons", and two "flange-plates", each of these seven pieces being continuous through the whole length of the beam. If the riveting is well done, the combination forms a single rigid beam whose safe load for a given span may be found by foregoing rules; in computing the moment of inertia, however, the portion of cross section cut out by the rivet holes must not be included. (This will be illustrated in a subsequent paragraph). The safe load having been computed from consideration of normal stresses only and the web being made

thick enough to take up the max. total shear, J_m, with safety, it still remains to design the riveting, through whose agency the web and flanges are caused to act together as a single continuous rigid mass. It will be on the side of safety to consider that at a given locality in the beam the shear carried by the rivets connecting the angles and flanges, per unit of length of beam, is the same as that carried by those connecting the angles and the web ("vertical stem plate"). The amount of this shear may be computed from the fact that it is equal to that occurring in the surface (parallel to the neutral surface) in which the web joins the flange, in case the web and flange were of continuous substance, as in a solid I-beam. But this shear must be of the same amount per horizontal unit of length in the vertical section of the web itself, where it joins the flange; (for from § 254 $Z = X$). But the shear in the vertical section of the web, being uniformly distributed, is the same per vertical linear unit at the junction with the flange as at any other part of the web section (§ 256), and the whole shear on the vertical section of web = J, the "total shear" of that section of the beam.

Hence we may state the following:

The riveting connecting the angles with the flanges (or the web with the angles) in any locality of a built beam, must safely sustain a shear equal to J ÷ height of web, per horizontal linear unit of the length of beam.

The strength of the riveting may be limited by the resistance of the rivet to being sheared (and this brings into account its cross section) or upon the crushing resistance of the side of the rivet hole in the plate (and this involves both the diameter of the rivet and the thickness of the metal in the web, flange, or angle). In its practice the N. J. Steel and Iron Co. allow 7500 lbs. per sq. inch shearing stress in the rivet (wrought iron), and 12500 lbs per sq. inch compressive resistance in the side of the rivet-hole, the axial plane section of the hole being the area of reference.

In fig. 259 the rivets connecting the web with the angles are in

double shear, which should be taken into account in considering their shearing strength, which is then double; those connecting the angles and the flange plates are in single shear. In fig. 260 (box beam) where the beam is built of two webs, four angles, and two flange plates, all the rivets are in single shear. If the web plate is very high compared with its thickness, vertical stiffeners in the form of T irons may need to be riveted upon them laterally [see §]

EXAMPLE. A built I-beam of wrought iron (see fig. 259) is to support a uniformly distributed load of 40 tons, its extremities resting on supports 20 feet apart, and the height and thickness of web being 20 ins. and $\frac{1}{2}$ in. respectively. How shall the rivets, which are $\frac{7}{8}$ in. in diameter, be spaced, between the web and the angles which are also $\frac{1}{2}$ in. in thickness. Referring to fig. 235 we find that $J = \frac{1}{2}W = 20$ tons at each support and diminished regularly to zero at the middle, where no riveting will therefore be required. (Units inch and pound). Near a support the riveting must sustain for each inch of length of beam a shearing force of ($J \div$ height of web) $= 40000 \div 20$ in. $= 2000$ lbs. Each rivet, having a sectional area of $\frac{1}{4}\pi\left(\frac{7}{8}\right)^2 = 0.60$ sq. inches, can bear a safe shear of $0.60 \times 7500 = 4500$ lbs. in single shear and ∴ of 9000 lbs. in double shear, which is the present case. But the safe compressive resistance of the side of the rivet hole in either the web or the angle is only $\frac{7}{8}$ in. $\times \frac{1}{2}$ in. $\times 12500 = 5470$ lbs., and thus determines the spacing of the rivets as follows:

2000 lbs. $\div 5470$ gives 0.30 as the number of rivets per inch of length of beam i.e. they must be $1 \div 0.30 = 3.3$ inches apart, centre to centre, near the supports; 6.6 inches apart at $\frac{1}{4}$ the span from a support; none at all in the middle.

However "the rivets should not be spaced closer than $2\frac{1}{2}$ times their diameter, nor further apart than 16 times the thickness of the plate they connect" is the rule of the N.J. Co.

As for the rivets connecting the angles and flange plates, being

in two rows and opposite (in pairs) the safe shearing resistance of a pair (each in single shear) is 9000 lbs., while the safe compressive resistance of the sides of the two rivet holes in the angle irons (the flange plate being much thicker) is = 10940 lbs.; Hence the former figure (9000) divided into 2000 lbs., gives 0.22 as the number of pairs of rivets per inch of length of the beam; i.e., the rivets in one row should be spaced 3.6 inches apart, centre to centre, near a support; the interval to be increased in inverse ratio to the distance from the middle of span, (bearing in mind the practical limitation just given.)

If the load is concentrated in the middle of the span, instead of uniformly distributed, J is constant along each half-span, (see fig. 234) and the rivet spacing must accordingly be made the same at all localities of the beam.

SPECIAL PROBLEMS IN FLEXURE.

258. DESIGNING CROSS SECTIONS OF BUILT BEAMS. The last paragraph dealt with the riveting of the various plates; we now consider the design of the plates themselves. Take for instance a built I-beam, fig. 261, one vertical stem-plate, four angle irons (each of sectional area = A, remaining after the holes are punched, with a gravity axis parallel to, and at a distance = a from its base), and two flange plates of width = b, and thickness = t. Let the whole depth of girder = h, and the diameter of a rivet hole = t′. To safely resist the tensile and compressive forces induced in this section by M_m inch-lbs. (M_m being the greatest moment in the beam which is prismatic) we must have from § 239.

$$M_m = \frac{R'I}{e} \qquad (1)$$

R′ for wrought iron = 12000 lbs. per sq. inch, e is $= \frac{1}{2}h$, while I, the moment of inertia of the compound section, is obtained as follows, taking into account the fact that the rivet holes cut out part of the material. In dealing with the sections of the angles and flanges, we consider them concentrated at their

centres of gravity (an approximation of course) and treat their moments of inertia about $\bar{N}$ as single terms in the series $\int dF z^2$ (see §85). The subtractive moments of inertia for the rivet holes in the web are similarly expressed.

$$\therefore \quad \begin{cases} I_N \text{ for web } = \frac{1}{12} b_0 (h-2t)^3 - 2b_0 t' \left[\frac{h}{2} - t - a'\right]^2 \\ I_N \text{ for four angles } = 4A\left[\frac{h}{2} - t - a\right]^2 \\ I_N \text{ for two flanges } = 2(b-2t')t\left(\frac{h-t}{2}\right)^2 \end{cases}$$

the sum of which makes the I of the girder. Eq. (1) may now be written

$$\frac{M_m h}{2R'} = I \qquad (2)$$

which is available for computing any one unknown quantity. The quantities concerned in I are so numerous and they are combined in so complex a manner that in any numerical example it is best to adjust the dimensions of the section to each other by successive assumptions and trials. The size of rivets need not vary much in different cases, nor the thickness of the web-plate, which as used by the N.J.Co. is "rarely less than $\frac{1}{4}$ or more than $\frac{5}{8}$ inch thick." The same Co. recommends the use of a single size of angle irons, viz.: $3'' \times 3'' \times \frac{1}{2}''$, for built girders of heights ranging from 12 to 36 inches, and also $\frac{3}{4}$ in. rivets, and gives tables computed from eq. (2) for the proportionate strength of each portion of the compound section.

EXAMPLE. (Units, inch and pound). A built I-beam with end supports, of span = 20 ft. = 240 inches, is to support a uniformly distributed load of 36 tons = 72000 lbs. If $\frac{3}{4}$ inch rivets are used, angle irons $3'' \times 3'' \times \frac{1}{2}''$, vertical web $\frac{1}{2}''$ in thickness, and plates 1 inch thick for flanges, how wide ($b = ?$) must these flange-plates be? taking $h = 22$ inches = total height of girder.

SOLUTION. From the table in §250 we find that the

max. M for this case is $\frac{1}{8}Wl$, where W = the total distributed load (including the weight of the girder) and l = span. Hence the left hand member of eq. (2) reduces to

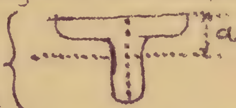
$$\frac{Wl}{16}\cdot\frac{h}{R'},=\frac{72000\times240\times22}{16\times12000}=1980$$

That is, the total moment of inertia of the section must be = 1980 biquad. inches, of which the web and angles supply a known amount, since $b_0=\frac{1}{2}$", $t=1$", $t'=\frac{3}{4}$", $a'=1\frac{3}{4}$", $A=2.0$ sq. in., $a=0.9$", and $h=22$", are known, while the remainder must be furnished by the flanges, thus determining their width b, the unknown quantity.

The effective area, A, of an angle iron is found thus: The full sectional area for the size given, $=3\times\frac{1}{2}+2\frac{1}{2}\times\frac{1}{2}$ = 2.75 sq. inches, from which deducting the two rivet holes we have

$$A=2.75-2\times\frac{3}{4}\times\frac{1}{2}=2.0 \text{ sq. in.}$$

The value $a=0.90$" is found by cutting out the shape of two angles from sheet iron, thus: and balancing it on a knife edge. (The gaps left by the rivet holes may be ignored, without great error, in finding a).



Hence, substituting we have

$$I_N \text{ for web} = \frac{1}{12}\cdot\frac{1}{2}\times20^3-2\times\frac{1}{2}\cdot\frac{3}{4}\left[8\tfrac{1}{4}\right]^2 = 282.3$$

$$I_N \text{ for four angles} = 4\times2\times[9.10]^2 = 662.5$$

$$I_N \text{ for two flanges} = 2(b-\tfrac{6}{4})\times1\times(10\tfrac{1}{2})^2 = 220.4\,(b-1.5)$$

$$\therefore 1980 = 282.3+662.5+(b-1.5)\,220.4$$

whence $b=4.6+1.5=6.1$ inches.

the required total width of each of the 1" flange plates. This might be increased to 6.5" so as to equal the united width of the two angles and web.

The rivet spacing can now be designed by § 257, and the assumed thickness of web, $\frac{1}{2}$", tested for the max. total shear by

§256. The latter test results as follows: The max. shear J_m occurs near either support and $= \frac{1}{2} W = 36000$ lbs. ∴, calling b_0' the least allowable thickness of web in order to keep the shearing stress as low as 4000 lbs. per sq. inch,

$$b_0' \times 20'' \times 4000 = 36000 \therefore b_0' = 0.45''$$

showing that the assumed width of $\frac{1}{2}$ in. is safe.

This girder will need vertical stiffeners near the ends, as explained subsequently, and is understood to be supported laterally. Built beams of double webs, or box-form, (see fig. 260) do not need this lateral support.

259. SET OF MOVING LOADS. When a locomotive passes over a number of parallel prismatic girders, each one of which experiences certain detached pressures corresponding to the different wheels, by selecting any definite position of the wheels on the span, we may easily compute the reactions of the supports, then form the shear diagram, and finally as in § 243 obtain the max moment, M_m, and the max. shear J_m, for this particular position of the wheels. But the values of M_m and J_m for some other position may be greater than those just found. We therefore inquire which will be the greatest moment among the infinite number of (M_m)'s (one for each possible position of the wheels on the span) It is evident from fig. 236, from the nature of the moment diagram, that when the pressures or loads are detached, the M_m for any position of the loads, which of course are in this case at fixed distances apart, must occur under one of the loads (i.e. under a wheel). We begin ∴ by asking: What is the position of the set of moving loads when the moment under a given wheel is greater than will occur under that wheel in any other position? For example, in fig. 262, in what position of the loads P_1, P_2, etc. on the span will the moment M_2, now, under P_2, be a maximum as compared with its value under P_2 in any other position on the span. Let R be the resultant of the loads <u>which are now on the span</u>, its <u>variable</u> distance from O be $= \bar{x}$, and its <u>fixed</u> distance from $P_2 = a'$; while a, b, c, etc. are

the fixed distances between the loads (wheets). For any values of $\bar{x}$, as the loading moves through the range of motion within which no wheel of the set under consideration goes off the span, and no new wheel comes on it, we have $R_1' = \frac{\bar{x}}{l} R$, and the moment under P_2

$$= M_2 = R_1 \left[l - (\bar{x} - a') \right] - P_3 b - P_4 (b + c)$$

$$\text{i.e.} \quad M_2 = \frac{R}{l}(l\bar{x} - \bar{x}^2 + a'\bar{x}) - P_3 b - P_4 (b + c) \quad \text{- - - - -} \quad (1)$$

In (1) we have M_2 as a function of $\bar{x}$, all the other quantities in the right hand member remaining constant as the loading moves; $\bar{x}$ may vary from $\bar{x} = a + a'$ to $\bar{x} = l - (c + b - a')$. For a max. M_2, we put $dM_2 \div d\bar{x} = 0$, i.e.

$$\frac{R}{l}(l - 2\bar{x} + a') = 0 \quad \therefore \bar{x} \text{ (for max. } M_2) = \tfrac{1}{2} l + \tfrac{1}{2} a'$$

(For this, or any other value of $\bar{x}$, $d^2 M_2 \div d\bar{x}^2$ is negative, hence a max. is indicated) For a max. M_2, then, R must be as far $(\frac{1}{2}a')$ on one side of the middle of the span as P_2 is on the other; i.e. as the loading moves, the moment under a given wheel becomes a max. when that wheel and the centre of gravity of all the loads <u>then on the span</u> are <u>equidistant from the middle of the span</u>.

In this way in any particular case we may find the respective max. moments occurring under each of the wheels during the passage and the greatest of these is the M_m to be used in the equation $M_m = R' I \div e$ for safe loading.

As to the shear J, for a given position of the wheels this will be greatest at one or the other support, and equals the reaction at that support. When the load moves toward either support the shear at that end of the beam evidently increases so long as no wheel rolls completely over and beyond it. To find J max., then, dealing with each support in turn, we compute the successive reactions at the support when the loading is successively so placed that consecutive wheels, in turn, are on the point of rolling off the girder at that end; the greatest of these is the max. shear, J_m. As the max. moment is apt to come under the heaviest load it may not be necessary to deal with more than one or two wheels in find-

ing M_m.

EXAMPLE. Given the following wheel pressures,

A← - - 8′ - - - - →B← - - - 5′ - - - →C← - - - 4′ - - - →D

4 tons 6 tons 6 tons 5 tons

on one rail which is continuous over a girder of 20 ft. span under a locomotive.

1. Required the position of the resultant of A, B, and C;
2. " " " " " " A, B, C, and D;
3. " " " " " " B, C, and D.
4. In what position of the wheels on the span will the moment under B be a max.? Ditto for wheel C. Required the values of these moments and which is M_m?
5. Required the value of J_m (max. shear), its location, and the position of loads.

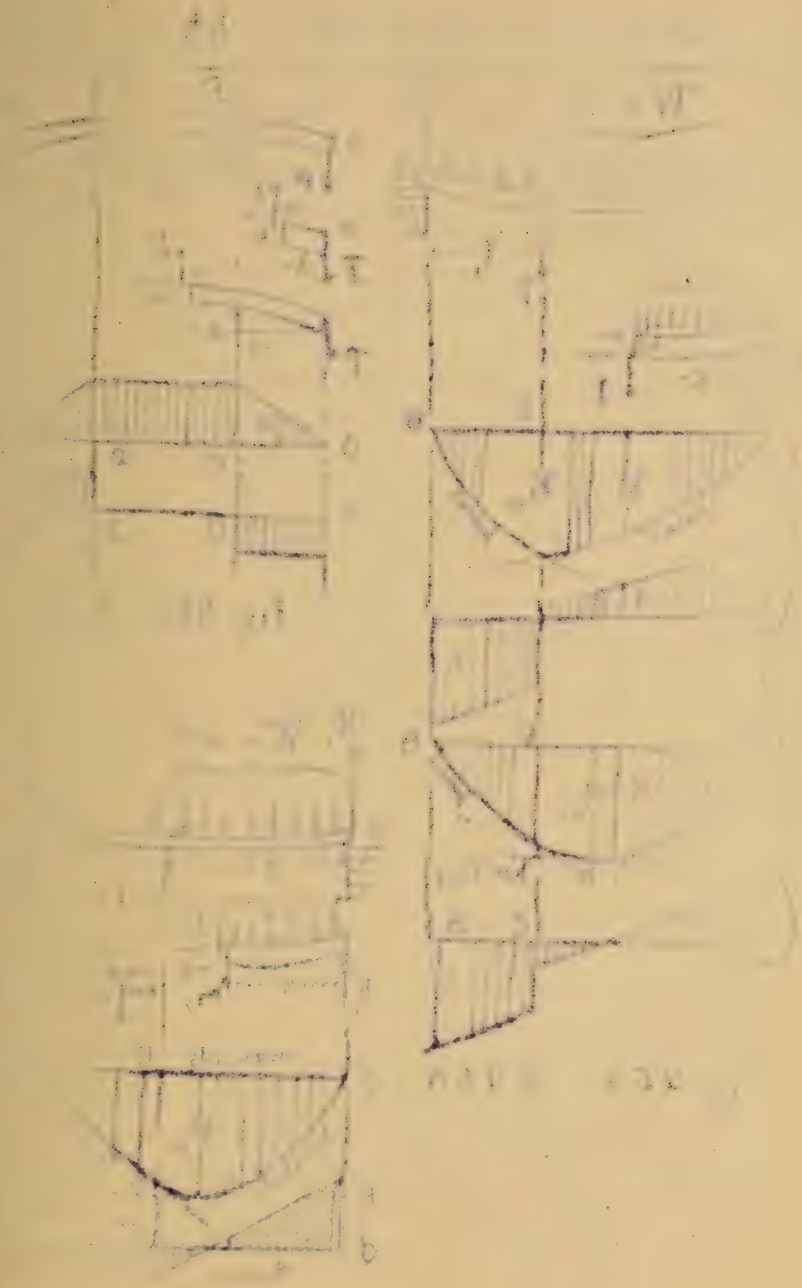
RESULTS. (1.) 7.8′ to right of A. (2.) 10′ to right of A. (3.) 4.4′ to right of B. (4.) Max. M_B = 1273000 inch-lbs. with all the wheels on; Max. M_C = 1440000 inch-lbs. with wheels B, C, and D on. (5.) J_m = 13.6 tons at right support with wheel D close to this support.

260. SINGLE ECCENTRIC LOAD. In the following special cases of prismatic beams, peculiar in the distribution of the loads, or mode of support, or both, the main objects sought are the values of the max. moment M_m, for use in the equation

$$M_m = \frac{R'I}{e} \quad (\text{see } \S 239);$$

and of the max. shear J_m, from which to design the web and riveting in the case of an I or box girder. The modes of support will be such that the reactions are independent of the form and material of the beam. As usual, the flexure is to be slight, and the forces are all perpendicular to the beam.

The present problem is that in fig. 263, the beam being prismatic, supported at the ends, with a single eccentric load, P. We shall first disregard the weight of the beam itself. Let the span $= l_1 + l_2$. First considering the whole beam free we have the reactions $R_1 = Pl_2 \div l$ and $R_2 = Pl_1 \div l$.



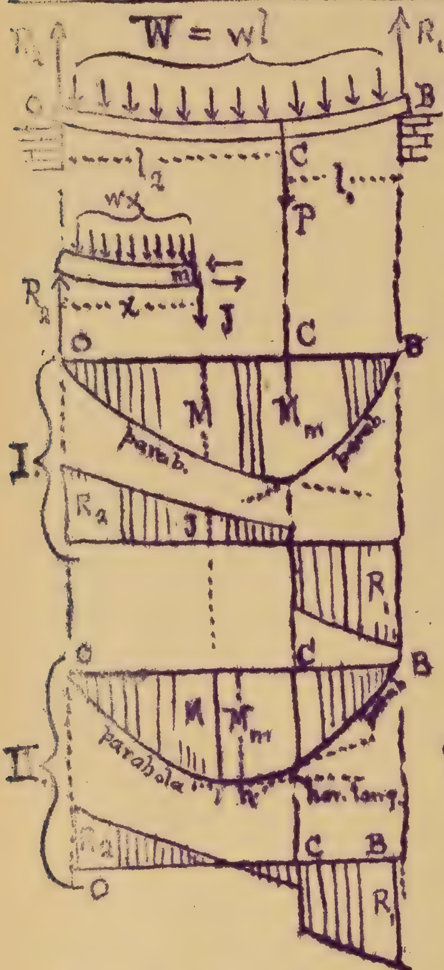


Fig. 264 § 260

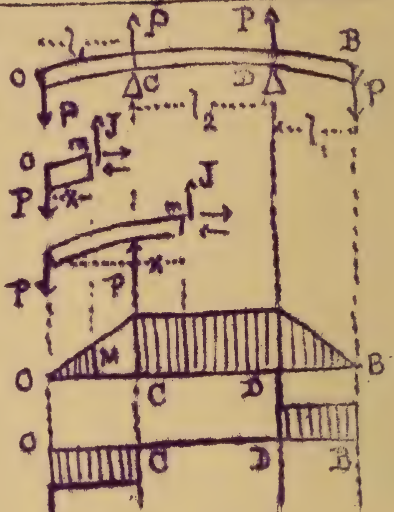


Fig. 265' § 261

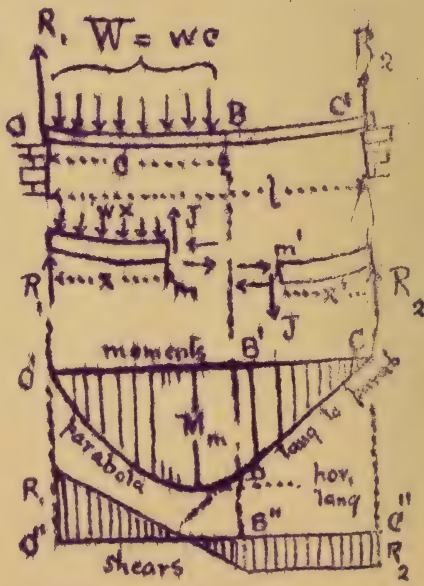


Fig. 266 § 262

Making a section at m and having Om free, x being $< l_2$, Σ(vert. compons.) $=0$ gives

$$R_2 - J = 0 \quad \text{i.e.} \quad J = R_2 \ ;$$

while from $\Sigma(\text{mom.})_m = 0$ we have

$$\frac{pI}{e} - R_2 x = 0 \quad \therefore M = R_2 x = \frac{Pl_1}{l} x$$

These values of J and M hold good between O and C, J being constant, while M is proportional to x. Hence for OC the shear diagram is a rectangle and the moment diagram a triangle. By inspection the greatest M for OC is for $x = l_2$, and $= Pl_1 l_2 \div l$. This is the max. M for the beam, since between C and B, M is proportional to the distance of the section from B.

$$\therefore M_m = \frac{Pl_1 l_2}{l} \quad \text{and} \quad \frac{R'I}{e} = \frac{Pl_1 l_2}{l} \qquad (1)$$

is the equation for safe loading.

$J = R_1$ in any section along CB and is opposite in sign to what it is on OC; i.e., practically, if a dove-tail joint existed anywhere on OC the portion of the beam on the right of such section would slide downwards relatively to the left hand portion; but vice versa on CB.

Evidently the max. shear $J_m = R_1$ or R_2, as l_2 or l_1 is the greater segment.

It is also evident that for a given span and given beam the safe load P', as computed from eq. (1) above, becomes very large as its point of application approaches a support; this would naturally be expected, but not without limit, as the shear for sections between the load and the support is equal to the reaction at the near support and may thus soon reach a limiting value, when the safety of the web, or the spacing of the rivets, if any, is considered.

Secondly, considering the weight of the beam, or any uniformly distributed loading, weighing W lbs. per unit of length of beam, in addition to P, Fig. 264, we have the reactions:

$$R_1 = \frac{Pl_2}{l} + \frac{W}{2}\ ; \text{ and } R_2 = \frac{Pl_1}{l} + \frac{W}{2}$$

Let l_2 be $> l_1$; then for a portion Om of length $x < l_2$, moments

about m give

$$\frac{pI}{e} - R_2 x + wx \cdot \frac{x}{2} = 0$$

i.e., on OC, $M = R_2 x - \frac{1}{2} wx^2$ - - - - - - - - (2)

Evidently for $x = 0$ (i.e. at O) $M = 0$, while for $x = l_2$ (i.e. at C) we have, putting $w = W \div l$

$$M_c = R_2 l_2 - \frac{1}{2} w l_2^2 = \frac{Pl_1 l_2}{l} + \frac{Wl_2}{2} - \frac{1}{2}\frac{Wl_2^2}{l} \quad \cdots (3)$$

It remains to be seen whether a value of M may not exist in some section between O and C, (i.e., for a value of $x < l_2$ in eq. (2)) still greater than M_c. Since (2) gives M as a continuous function of x between O and C, we put $dM \div dx = 0$, and obtain, substituting the value of the constants R_2 and w,

$$R_2 - wx = 0 \quad \therefore \; x_n \left\{\text{for } M \begin{smallmatrix}\text{max.}\\ \text{or}\\ \text{min.}\end{smallmatrix}\right\} = \frac{Pl_1}{W} + \frac{1}{2} l \quad \cdot\bullet\cdot \; (4)$$

This must be for M max., since $d^2M \div dx^2$ is negative when this value of x is substituted. If the particular value of x given by (4) is $< l_2$, the corresponding value of M (call it M_n) from eq. (2) will occur on OC and will be greater than M_c (Diagrams II in fig. 264 show this case); but if x_n is $> l_2$, we are not concerned with the corresponding value of M, and the greatest M on OC would be M_c.

For the short portion BC, which has moment and shear diagrams of its own not continuous with those for OC, it may easily be shown that M_c is the greatest moment of any section. Hence the M max, or M_m, of the whole beam is either M_c or M_n, according as x_n is $>$ or $< l_2$. This latter criterion may be expressed thus [with $l_2 - \frac{1}{2} l$ denoted by l_3, the distance of P from the middle of the span]:

From eq. (4) $\left[\left(\frac{Pl_1}{W} + \frac{1}{2} l\right) \gtrless l_2\right]$ is equiv. to $\left[\frac{P}{W} \gtrless \frac{l_3}{l_1}\right]$

and since from (4) and (2)

$$M_n = \left[\frac{Pl_1}{l} + \frac{1}{2} W\right]\left[\frac{Pl_1}{W} + \frac{1}{2} l\right] - \frac{1}{2}\frac{W}{l}\left[\frac{Pl_1}{W} + \frac{1}{2} l\right]^2 \ldots\ldots (5)$$

The equation for safe loading is

$$\left.\begin{aligned} \frac{R'I}{e} &= M_c\,, \text{ when } \frac{P}{W} \text{ is } > \frac{l_3}{l_1} \\ \text{and} \quad \frac{R'I}{e} &= M_n\,, \text{ when } \frac{P}{W} \text{ is } < \frac{l_3}{l_1} \end{aligned}\right\} \quad \ldots\ldots \quad (6)$$

See eqs. (3) and (5) for M_c and M_n

If either P, W, l_3, or l_1 is the unknown quantity sought, the criterion of (6) can not be applied and we ∴ use both equations in (6) and the discriminate between the two results.

The greatest shear is $J_m = R_1$, in Fig. 264, where l_2 is $> l_1$.

261. TWO EQUAL TERMINAL LOADS. TWO SYMMETRICAL SUPPORTS. Fig. 265. [Same case as in Fig. 231, § 238] Neglect weight of beam. The reaction at each support being $= P$, (from symmetry), we have for a free body Om with $x < l_1$

$$Px - \frac{pI}{e} = 0 \;\therefore M = Px \qquad (1);$$

while where $x > l_1$ and $< l_1 + l_2$

$$Px - P(x - l_1) - \frac{pI}{e} = 0 \;\therefore M = Pl_1 \quad --- (2)$$

That is, see (1), M varies directly with x between O and C, while between C and B it is constant. Hence for safe loading

$$\frac{R'I}{e} = M_m \quad \text{i.e.} \quad \frac{R'I}{e} = Pl_1 \quad ---- (3)$$

The construction of the moment diagram is evident from equations (1) and (2)

As for J, the shear, the same free bodies give, from Σ (vert. forces) $= 0$,

$$\text{On } OC \;----\; J = P \qquad (4)$$

$$\text{On } CD \;----\; J = P - P = \text{zero} \qquad (5)$$

(4) and (5) might also be obtained from (1) and (2) by writing $J = dM \div dx$, but the former method is to be preferred in most cases since the latter requires M to be expressed as a function of x while the former is applicable for examining separa

sections without making use of a variable.)

If the beam is an I-beam, the fact that J is zero anywhere on OC would indicate that we may dispense with a web along OC to unite the two flanges; but the lower flange being in compression and forming a "long column" would tend to buckle out of a straight line if not stayed by a web connection with the other, or some equivalent bracing.

262. UNIFORM LOAD OVER PART OF THE SPAN. TWO END SUPPORTS. Fig. 266. Let the load $= W$, extending from one support over a portion $= c$, of the span, (on the left, say), so that $W = wc$, w being the load per unit of length. Neglect weight of beam. For a free body Om of any length $x < \overline{OB}$ (i.e. $< c$), Σ moms$_m = 0$ gives

$$\frac{pI}{e} + \frac{wx^2}{2} - R_1 x = 0 \quad \therefore M = R_1 x - \frac{wx^2}{2} \quad \text{- - - - (1)}$$

which holds good for any section on OB. As for sections on BC it is more simple to deal with the free body $m'C$, of length $x' < CB$ from which we have $M = R_2 x'$ - - - - - - - - (2)

which shows the moment curve for BC to be a straight line $D'C'$, tangent at D to the parabola $O'D$ representing eq. (1) (If there were a concentrated load at B, $C'D$ would meet the tangent at D at an angle instead of co-inciding with it; let the student show why, from the shear diagram.)

The shear for any value of x on OB is:

$$\text{On } OB \text{ - - - - } J = R_1 - wx \qquad (3)$$

$$\text{while on } BC \text{ - - - - } J = R_2 = \text{constant} \text{ - - - - } (4)$$

The shear diagram is constructed accordingly.

To find the position of the max. ordinate of the parabola, (and this from previous statements concerning the tangent at the point D must occur on OB, as will be seen, and will be the M_m for the whole beam) we put $J = 0$ in eq. (3) whence

$$x \text{ (for } M_m) = \frac{R_1}{w} = \frac{W}{w} \cdot \frac{[l - \frac{c}{2}]}{l} = c - \frac{c^2}{2l}$$

and is less than c, as expected. The value of $R_1 = W(l - \frac{c}{2}) \div l$

Figs. 267 to 271 To face p. 91 §§ 263 to 268

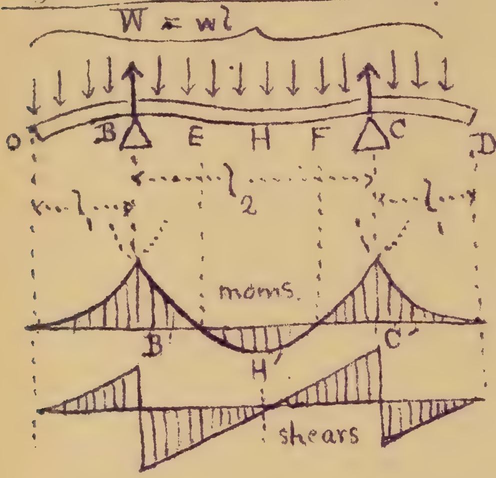


Fig. 267 § 263

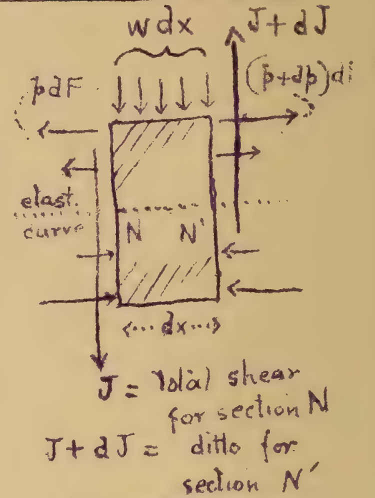


Fig. 269 § 266

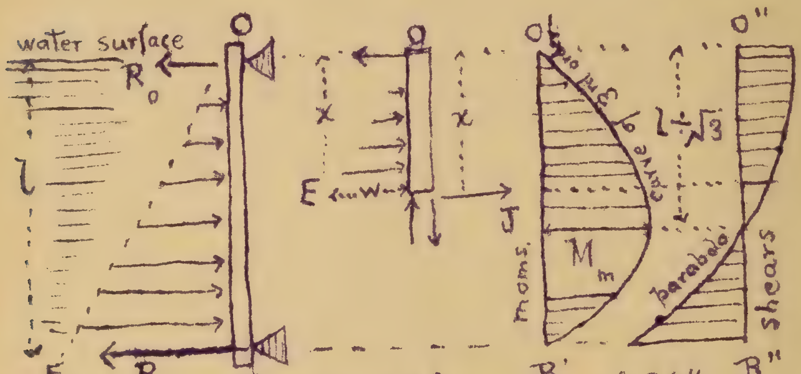


Fig. 268 § 264

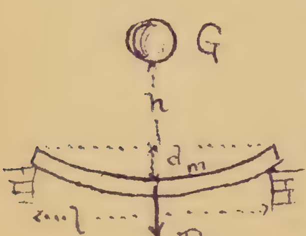


Fig. 270 P_m § 267

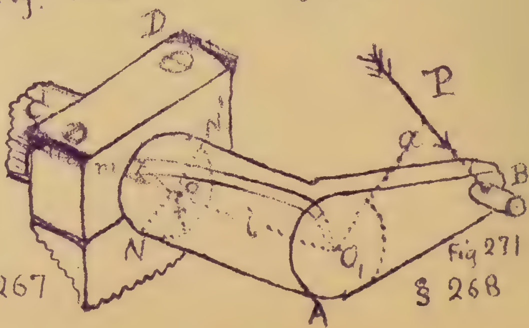


Fig 271 § 268

$=(wc \div l)(l-\frac{c}{2})$, (the whole beam free) has been substituted. This value of x substituted in eq. (1) gives

$$M_m=\left(1-\frac{1}{2}\frac{c}{l}\right)^2\cdot\frac{1}{2}\cdot Wc \;\therefore\; \frac{R'I}{e}=\frac{1}{2}\left[1-\frac{1}{2}\cdot\frac{c}{l}\right]^2 Wc$$

is the equation for safe loading.

The max. shear J_m is found at O and is $= R_1$, which is evidently $> R_2$, at C.

263. UNIFORM LOAD OVER WHOLE LENGTH WITH TWO SYMMETRICAL SUPPORTS. Fig. 267. With the notation expressed in the figure, the following results may be obtained, after having divided the length of the beam into three parts for separate treatment as necessitated by the external forces, which are the distributed load W and the two reactions, each $=\frac{1}{2}W$. The moment curve is made up of parts of three distinct parabolas, each with its axis vertical. The central parabola may sink below the horizontal axis of reference if the supports are far enough apart, in which case (see fig.) the elastic curve of the beam itself becomes concave upward between the points E and F of "contrary flexure". At each of these points the moment must be zero, since the radius of curvature is ∞ and $M = EI \div \rho$ (see § 231) at any section; that is, at these points the moment curve crosses its horizontal axis.

As to the location and amount of the max. moment M_m, inspecting the diagram we see that it will be either at H' the middle, or at both of the supports B' and C' (which from symmetry have equal moments), i.e.

$$M_m\left[\text{and} \therefore \frac{R'I}{e}\right]=\begin{cases}\text{either } \dfrac{W}{2l}\left[\frac{1}{4}l_2^2-l_1^2\right] \text{ - - - at H}\\ \text{or } \quad \dfrac{W}{2l}\,l_1^2 \text{ - - - at B and C}\end{cases}$$

according to which is the greater in any given case; i.e. according as l is $>$ or $< l_1\sqrt{8}$.

The shear close on the left of $B = wl_1$, while close to the right of B it $=\frac{1}{2}W - wl_1$

Hence $J_m = \left\{ \begin{matrix} wl_1 \\ \frac{1}{2}W - wl_1 \end{matrix} \right\}$ according as $l_1 \gtrless \frac{1}{4}l$

264. HYDROSTATIC PRESSURE AGAINST A VERTICAL PLANK.

From elementary hydrostatics we know that the pressure, per unit area, of quiescent water against the vertical side of a tank, varies directly with the depth, x, below the surface, and equals the weight of a prism of water whose altitude $= x$, and whose sectional area is unity. See fig. 268. The plank is of rectangular cross section, its constant breadth $= b$ being ⊤ to the paper, and receives no support except at its two extremities O and B, O being level with the water surface. The loading, or pressure, per unit of length of the beam, is here variable and, by above definition, is $= w = \gamma x b$, where γ = weight of a cubic unit (i.e. the heaviness, see § 7) of water, and $x = Om$ = depth of any section m below the surface. The hydrostatic pressure on $dx = w\,dx$. These pressures for equal dx's, vary as the ordinates of a triangle OE_1B.

Consider Om free. Besides the elastic forces of the exposed section m, the forces acting are the reaction R_0, and the triangle of pressure OEm. The total of the latter is

$$W_x = \int_0^x w\,dx = \gamma b \int_0^x x\,dx = \gamma b \frac{x^2}{2} \quad \text{------} \quad (1)$$

and the sum of the moments of these pressures about m is equal to that of their resultant (= their sum, since they are parallel) about m, and $\therefore = \gamma b \frac{x^2}{2} \cdot \frac{x}{3}$.

[From (1) when $x = l$, we have for the total water pressure on the beam $W_l = \gamma b \frac{l^2}{2}$ and since one third of this will be borne at O we have $R_0 = \frac{1}{6}\gamma b l^2$]

Now putting Σ (moms. about the neutral axis of m) $= 0$, for Om free, we have

$$R_0 x - W_x \cdot \frac{x}{3} - \frac{pI}{e} = 0 \quad \therefore M = \frac{1}{6}\gamma b (l^2 x - x^3) \quad \text{-} \quad (2)$$

(which holds good from $x = 0$ to $x = l$). From Σ (horiz. forces)

$=0$ we have also the shear

$$J = R_0 - W_x = \frac{1}{6}\gamma b(l^2 - 3x^2) \quad \text{---} \quad (3)$$

as might also have been obtained by differentiating (2), since $J = dM \div dx$. By putting $J=0$ (§240, corollary) we have for a max. M, $x = l \div \sqrt{3}$, which is less than l and hence is applicable to the problem. Substitute this in eq. 2, and reduce, and we have

$$\frac{R'I}{e} = M_m\text{, i.e. } \frac{R'I}{e} = \frac{1}{9}\cdot\frac{1}{\sqrt{3}}\cdot\gamma b l^3 \quad \text{----} \quad (4)$$

as the equation for safe loading.

265. EXAMPLE. If the thickness of the plank is h, required $h=?$, if R' is taken $=1000$ lbs. per sq. in. for timber (§251), and $l = 6$ feet. For the <u>inch</u>-<u>pound</u>-<u>second</u> system of units, we must substitute $R'=1000$; $l=72$ <u>inches</u>; $\gamma = 0.036$ lbs. per cubic inch (heaviness of water in this system of units); while $I = bh^3 \div 12$, (§247), and $e = \frac{1}{2}h$. Hence from (4) we have

$$\frac{1000\,bh^3}{12 \times \frac{1}{2}h} = \frac{0.036\,b \times 72^3}{9\sqrt{3}} \therefore h^2 = 5.16 \therefore h = 2.27 \text{ in.}$$

It will be noticed that since x for $M_m = l \div \sqrt{3}$, and not $\frac{2}{3}l$, M_m does not occur in the section opposite the resultant of the water pressure; see fig. 268. The shear curve is a parabola here; eq. (3).

266. THE FOUR x-DERIVATIVES OF THE ORDINATE OF THE ELASTIC CURVE. If $y = $ func. (x) is the equation of the elastic curve for any portion of a loaded beam, on which portion the load per unit of length of the beam is $w=$ either zero, (Fig. 234) or $=$ constant, (Fig. 235), or $=$ a continuous func. (x) (as in the last §), we may prove, as follows, that $w =$ the x-derivative of the shear. Fig. 269. Let II and II' be two consecutive cross-sections of a loaded beam, and let the block between them, bearing its portion, $w\,dx$, of a distributed load, be considered free. The elastic forces consist of the two stress-couples (tensions

and compressions) and the two shears, J and $J + dJ$, dJ being the shear-increment consequent upon x receiving its increment dx. By putting Σ (vert. components) $= 0$ we have

$$J + dJ - wdx - J = 0 \quad \therefore \; w = \frac{dJ}{dx}$$

Q.E.D. But J itself $= dM \div dx$, (§ 240) and $M = [d^2y \div dx^2]\, EI$. By substitution, then, we have the following relations

y = func. (x) = ordinate at any point of the elastic curve (1)

$\frac{dy}{dx} = \alpha =$ slope " " " " " " (2)

$EI\frac{d^2y}{dx^2} = M$ = ordinate (to scale) of the moment curve (3)

$EI\frac{d^3y}{dx^3}$ = the shear, J = { the ordinate (to scale) of the shear diagram } .... (4)

$EI\frac{d^4y}{dx^4} = w$ = { the load per unit of length of beam = ordinate (to scale) of a curve of loading } .. (5)

If, then, the equation of the elastic curve (the neutral line of the beam itself; a reality, and not artificial like the other curves spoken of) is given; we may by successive differentiation, for a prismatic and homogeneous beam so that both E and I are constant, find the other four quantities mentioned.

As to the converse process, (i.e. having given w as a function of x, to find expressions for J, M and y as functions of x) this is more difficult, since in taking the x-anti-derivative each time an unknown constant must be added and determined. The case just treated in § 264, however, offers a very simple case since w is the same function of x, along <u>the whole beam</u> and there is therefore but one elastic curve to be determined.

We $\therefore$ begin, numbering backward, with

$$EI\frac{d^4y}{dx^4} = -\gamma bx \quad \left\{\text{since } w = \gamma bx \text{ see last § and fig. 268}\right\} \cdots (5a)$$

[N.B. This derivative ($dJ \div dx$) is negative since dJ and dx have contrary signs]

$$\therefore (\text{shear} =) EI\frac{d^3y}{dx^3} = -\gamma b\frac{x^2}{2} + \text{Const.}$$

But writing out this equation for $x=0$, i.e. for the point O, where the shear $= R_0$ we have $R_0 = 0 + \text{Const.}$ $\therefore \text{Const} = R_0$ and hence write

$$EI\frac{d^3y}{dx^3} = -\gamma b\frac{x^2}{2} + R_0 \quad \text{- - - - (shear) - - - (4a)}$$

Again taking the x-anti-derivative of both sides

$$(\text{Moment} =) EI\frac{d^2y}{dx^2} = -\gamma b\frac{x^3}{6} + R_0 x + (\text{Const.} = 0) \quad \text{- - - (3a)}$$

[At O, $x=0$ and $M=0$ $\therefore$ Const. $=0$] Again,

$$EI\frac{dy}{dx} = -\gamma b\frac{x^4}{24} + R_0\frac{x^2}{2} + C'$$

At O, where $x=0$, $dy \div dx = \alpha_0$ the unknown slope of the elastic line at O, and hence $C' = EI\alpha_0$

$$\therefore (EI \times \text{slope}) EI\frac{dy}{dx} = -\gamma b\frac{x^4}{24} + R_0\frac{x^2}{2} + EI\alpha_0 \quad \text{- - - (2a)}$$

Passing now to y itself, and remembering that at O, both y and x are zero, so that the constant, if added, would $=$ zero, we obtain (inserting the value of R_0 from last §)

$$EIy = -\gamma b\frac{x^5}{120} + \gamma b l^2\frac{x^3}{36} + EI\alpha_0 x \quad \text{- - - - - (1a)}$$

the equation of the elastic curve. This, however, contains the unknown constant α_0 = the slope at O. To determine α_0 write out eq. (1a) for the point B, fig. 268, where x is known to be equal to l, and y to be $=$ zero, solve for α_0, and insert its value both in (1a) and (2a). To find the point of max. y (i.e. of greatest deflection) in the elastic curve, write the slope, i.e. $dy \div dx$, $=$ zero [see eq. (2a)] and solve for x; four values will be obtained, of which the one lying between O and l is obviously the one to be taken. This value of x substituted in (1a) will give the max. deflection. The location of this max. deflection is neither under the centre of action of the

load $(x = \frac{2}{3} l)$, nor at the section of max. moment $(x = l \div \sqrt{3})$.

The qualities of the left hand members of equations (1) to (5) should be carefully noted. E.g. in the inch-pound-second system of units we should have:

1. y (a linear quantity) = (so many) inches.
2. $dy \div dx$ (an abstract number) = (so many) abstract units.
3. M (a moment) = (so many) inch-pounds.
4. J (a shear, i.e., force) = (····) pounds
5. w (force per linear unit) = (····) {pounds per running inch of beam's length.

As to the quantities E, and I, individually, E is pounds per sq. in., and I has four linear dimensions, i.e. (so many) biquadratic inches.

267. RESILIENCE OF BEAM WITH END SUPPORTS.

Fig. 270. If a mass whose weight is G be allowed to fall freely through a height $= h$ upon the centre of a beam supported at its extremities, the pressure P felt by the beam increases from zero at the first instant of contact up to a maximum P_m, as already stated in § 233 a, in which the equation was derived, d_m being small compared with h,

$$Gh = \frac{1}{96} \cdot \frac{P_m^2 l^3}{EI} \quad \cdots\cdots \quad (a)$$

The elastic limit is supposed not passed. In order that the max. normal stress in any outer fibre shall at most be $= R'$, a safe value, (see table § 251) we must put $\frac{R'I}{e} = \frac{P_m l}{4}$ [according to eq. (2) § 241,] i.e. in equation (a) above; substitute $P_m = [4R'I] \div le$, which gives

$$Gh = \frac{1}{6} \cdot \frac{R'^2 I l}{E e^2} = \frac{1}{6} \cdot \frac{R'^2}{E} \cdot \frac{k^2}{e^2} \cdot Fl = \frac{1}{6} \cdot \frac{R'^2}{E} \cdot \frac{k^2}{e^2} V \quad \cdots \quad (b)$$

having put $I = Fk^2$ (k being the radius of gyration (§ 85)) and $Fl = V$ the volume of the (prismatic) beam. From equation (b) we have the energy, Gh, (in ft. lbs., or inch-lbs.) of the vertical blow at its middle which the beam of fig. 270 will safely bear, and any one unknown quantity can be computed from

it, but the mass of G should not be small compared with that of the beam)

The energy of this safe impact, for two beams of the same material and similar cross-sections (similarly placed), is seen to be proportional to their volumes; while if furthermore their cross sections are the same and similarly placed, the safe Gh is proportional to their lengths. (These same relations hold good, approximately, beyond the elastic limit.)

It will be noticed that the last statement is just the converse of what was ~~formed~~ found in §245 for static loads, (the pressure at the centre of the beam being then equal to the weight of the safe load); for there the longer the beam (and ∴ the span) the less the safe load, in inverse ratio. As appropriate in this connection a quotation will be given from "The Strength of Materials and Structures", by Sir John Anderson, London, 1884, viz: (p. 186)

"It appears from the published experiments and statements of the Railway Commissioners, that a beam 12 ft. long will only support $\frac{1}{2}$ of the steady load that a beam 6 ft. long of the same breadth and depth will support, but that it will bear double the weight suddenly applied, as in the case of a weight falling upon it", (from the same height, should be added); "or if the same weights are used, the longer beam will not break by the weight falling upon it unless it falls through twice the distance required to fracture the shorter beam."

268. COMBINED FLEXURE AND TORSION. CRANK SHAFTS. Fig. 271. Let AB be the crank, and NA the portion projecting beyond the nearest bearing N. P is the pressure of the connecting-rod against the crank-pin at a definite instant, the rotary motion being uniform. Let a = the perpendicular dropped from the axis OO_1 of the shaft upon P, and l = the distance of a, along the axis OO_1, from the cross section NmN' of the shaft, close to the bearing. Let NN' be a diameter of this section and parallel to a. In considering the portion NO_1B free, and thus exposing the circular section NmN',

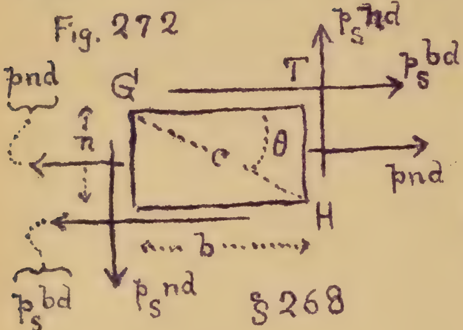


Fig. 272 § 268

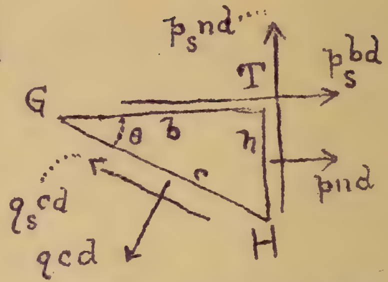


Fig. 273 § 268

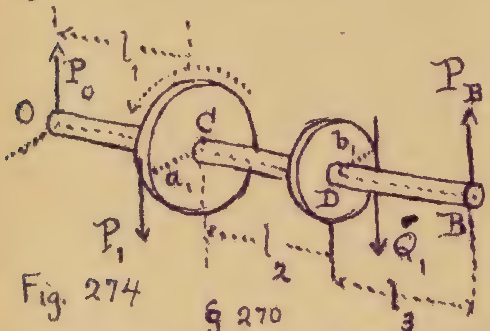


Fig. 274 § 270

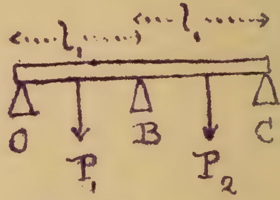


Fig. 274a. § 271

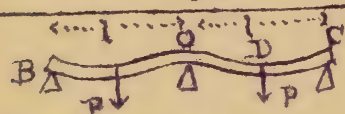


Fig. 275 § 272

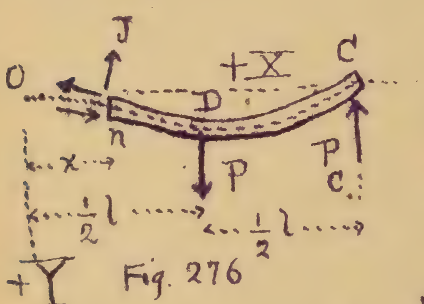


Fig. 276

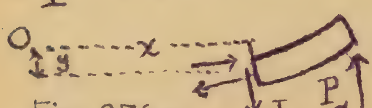


Fig. 276a.

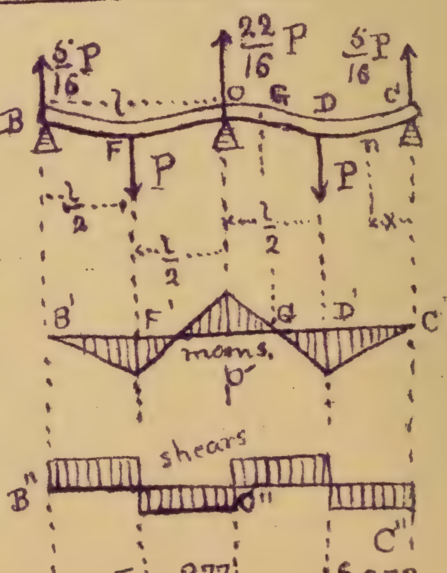


Fig. 277 § 272

we may assume that the stresses to be put in on the elements of this surface are the tensions (above NN') and compressions (below NN') and shears ⅂ to NN', due to the bending action of P; and the shearing stress tangent to the circles which have O as a common centre, and pass through the respective dF's or elementary areas, these latter stresses being due to the twisting action of P.

In the former set of elastic forces let p = the tensile stress per unit of area in the small parallelopipedical element m of the helix which is furthest from NN' (the neutral axis) and I = the moment of inertia of the circle about NN'; then taking moments about NN' for the free body (disregarding the motion) we have as in cases of flexure (see §239)

$$\frac{pI}{r} = Pl \text{; i.e., } p = \frac{Plr}{I} \quad \text{- - - - - - - - - (a)}$$

[None of the shears has a moment about NN'] Next taking moments about OO_1, (the flexure elastic forces, both normal and shearing, having no moments about OO_1) we have as in torsion (§216)

$$\frac{p_s I_p}{r} = Pa \text{; i.e., } p_s = \frac{Par}{I_p} \quad \text{- - - - - - - - - (b.)}$$

in which p_s is the shearing stress per unit of area, in the torsional elastic forces, on any outer most dF, as at m; and I_p the polar moment of inertia of the circle about its centre O.

Next consider free, in fig. 272, a small parallelopiped taken from the helix at m (of fig. 271). The stresses [see §209] acting on the four faces ⊥ to the paper in fig. 272 are there represented, the dimensions (infinitesimal) being n ∥ to NN', b ∥ to OO_1, and d ⊥ to the paper in fig. 272. By altering the ratio of b to n we may make the angle θ what we please. It is now proposed to consider free the triangular prism GHT, to find the intensity of normal stress q, per unit of area, on the diagonal plane GH, (of length = c) which is a bounding face of that triangular prism. See fig. 273. By writing Σ

Σ(compons. in direction of normal to GH) = 0, we shall have, transposing,

$$qcd = pnd\sin\theta + p_s bd\sin\theta + p_s nd\cos\theta;$$

and solving for q

$$q = p\frac{n}{c}\sin\theta + p_s\left[\frac{b}{c}\sin\theta + \frac{n}{c}\cdot\cos\theta\right]; \quad \ldots\ldots (1)$$

but $n : c = \sin\theta$ and $b : c = \cos\theta$ $\therefore$

$$q = p\sin^2\theta + p_s\, 2\sin\theta\cos\theta \quad \ldots\ldots \quad (2)$$

This may be written (see eqs. 63 and 60, O. W. J. Trigonometry)

$$q = \tfrac{1}{2}p(1 - \cos 2\theta) + p_s\sin 2\theta \quad \ldots\ldots \quad (3)$$

As the diagonal plane GH is taken in different positions (i.e., as θ varies), this tensile stress q (lbs. per sq. in. for instance) also varies, being a function of θ, and its max. value may be $> p$. To find θ for q max. we put

$$\frac{dq}{d\theta} = 0; \quad \text{i.e. } p\sin 2\theta + 2p_s\cos 2\theta = 0 \ldots\ldots (4)$$

and obtain:

$$\tan\left[2(\theta \text{ for } q \text{ max})\right] = -\frac{2p_s}{p} \quad \ldots\ldots (5)$$

Call this value of θ, θ'. Since $\tan 2\theta'$ is negative, $2\theta'$ lies either in the second or fourth quadrant, and hence

$$\sin 2\theta' = \pm\frac{2p_s}{\sqrt{p^2 + 4p_s^2}} \quad \text{and} \quad \cos 2\theta' = \mp\frac{p}{\sqrt{p^2 + 4p_s}} \quad (6)$$

[See equations 28 and 29 Trigonometry, O.W.J.] The upper signs refer to the second quadrant, the lower to the fourth. If we now differentiate (4), obtaining

$$\frac{d^2q}{d\theta^2} = 2p\cos 2\theta - 4p_s\sin 2\theta \quad - - - - - (7)$$

we note that if the sine and cosine of the $[2\theta']$ of the 2nd quadrant [upper signs in (6)] are substituted in (7) the result is <u>negative</u>, indicating a maximum; that is, q is a maximum for θ = the θ' of eq. (6) <u>when the upper signs are taken</u> (2nd

quadrant). To find q max., then put θ' for θ in (3) substituting from (6) (upper signs). We thus find

$$q \max = \frac{1}{2}\left[p + \sqrt{p^2 + 4p_s^2}\right] \text{-------} (8.)$$

A similar process, taking components parallel to GH, Fig. 273, will yield q_s max., i.e. the max. shear per unit of area, which for a given p and p_s exists on the diagonal plane GH in any of its possible positions, as θ varies. This max. shearing stress is

$$q_s \max = \frac{1}{2}\sqrt{p^2 + 4p_s^2} \text{ .... } (9)$$

In the element diametrically opposite to m in Fig. 271, p is compression instead of tension; q max will also be compression but is numerically the same as the q max. of eq. 8

EXAMPLE. In Fig. 271 suppose P = 2 tons = 4000 lbs., a = 6 in., l = 5 in., and that the shaft is of wro't iron. Required its radius that the max. tension or compression may not exceed R' = 12000 lbs. per sq. in.; nor the max. shear exceed S' = 7000 " " " ". That is put q = 12000 in eq. (8) and solve for r: also put q_s = 7000 in (9) and solve for r. The greater value of r should be taken. From equations (a) and (b) we have (see §§ 219 and 247 for I_p and I)

$$p = \frac{4Pl}{\pi r^3} \quad \text{and} \quad p_s = \frac{2Pa}{\pi r^3}$$

which in (8) and (9) give

$$\max. q = \frac{1}{2}\frac{P}{\pi r^3}\left[4l + \sqrt{(4l)^2 + 4(2a)^2}\right] \text{--} (8a)$$

and

$$\max. q_s = \frac{1}{2}\frac{P}{\pi r^3}\sqrt{(4l)^2 + 4(2a)^2} \text{ - - - } (9a)$$

With max. q = 12000, and the values of P, a, and l, already given, (units, inch and pound) we have from (8a), $r^3 = 2.72$ cubic inches ∴ r = 1.39 inches

Next, with max. q_s = 7000; P, a, and l as before; from (9a) $r^3 = 2.84$ cub. inches ∴ r = 1.41 inches

The latter value of r, 1.41 inches, should be adopted.

It is here supposed that the crank-pin is in such a position (when $P = 4000$ lbs. and $a = 6$ in.) that q_1 max. (and q_s max.) are greater than for any other position; a number of trials may be necessary to decide this, since P and a are different with each new position of the connecting-rod. If the shaft and its connections are exposed to shocks, R' and S' should be taken much smaller.

270. ANOTHER EXAMPLE of combined torsion and flexure is shown in fig. 274. The work of the working force P (vertical cog-pressure) is expended in overcoming the resistance (another vertical cog-pressure) Q.

That is, the rigid body consisting of the two wheels and shaft is employed to transmit power, at a uniform angular velocity, and since it is symmetrical about its axis of rotation the forces acting on it, considered free, form a balanced system. (See § 114) Hence given P and the various geometrical quantities a, b, etc., we may obtain Q_1, and the reactions P_0 and P_B, in terms of P_1. The greatest moment of flexure in the shaft will be either $P_0 l_1$, at C; or $P_B l_3$, at D. The portion CD is under torsion, of a moment of torsion $= P_1 a_1 = Q_1 b_1$. Hence we proceed as in the example of § 269, simply putting $P_0 l_1$ (or $P_B l_3$, whichever is the greater) in place of Pl, and $P_1 a_1$ in place of Pa. We have here neglected the weight of the shaft and wheels. If Q_1 were an upward vertical force and hence on the same side of the shaft as P_1, the reactions P_0 and P_B would be less than before, and one or both of them might be reversed in direction.

CHAP. IV.

FLEXURE, CONTINUED.

CONTINUOUS GIRDERS.

271. DEFINITION. A continuous girder, for present

purposes, may be defined to be a loaded strait beam supported in <u>more than two points</u>, in which case we can no longer, as heretofore, determine the reactions at the supports from simple Statics alone, but must have recourse to the equations of the several elastic curves formed by its neutral line, which equations involve directly or indirectly the reactions sought; the latter may then be found as if they were constants of integration. Practically this amounts to saying that the reactions depend on the manner in which the beam bends; whereas in previous cases, with only two supports, the reactions were independent of the forms of the elastic curves (the flexure being slight, however).

As an ILLUSTRATION, if the straight beam of fig. 274 is placed on three supports O, B, and C, at the same level, the reactions of these supports seem at first sight indeterminate; for on considering the whole beam free, we have three unknown quantities and only two equations, viz: Σ (vert. compons) = 0 and Σ (moms. about some point) = 0. If now B be gradually lowered, it receives less and less pressure, until it finally reaches a position where the beam barely touches it; and then B's reaction is zero and O and C support the beam as if B were not there. As to how low B must sink to attain this position, depends on the stiffness and load of the beam. Again, if B be raised above the level of O and C it receives greater and greater pressure, until the beam fails to touch one of the other supports. Still another consideration is that if the beam were tapering in form, being stiffest at B, and pointed at O and at C, the three reactions would be different from what they would be if the beam were prismatic. It is therefore evident that for more than two supports the values of the reactions depend on the relative heights of the supports and upon the form and elasticity of the beam. The circumstance that the beam is made <u>continuous</u> over the support B, instead of being cut apart at B into two independent beams, each covering its own span and having its own two supports, shows the significance of the term "con-

tinuous girder". The case in Fig. 271 is approximately that of a drawbridge, closed.

All the cases here considered will be comparatively simple, from the symmetry of their conditions. The beams will all be prismatic, and all external forces (i.e. loads and reactions) perpendicular to the beam and in the same plane. All supports at the SAME LEVEL.

272. TWO EQUAL SPANS; TWO CONCENTRATED LOADS, ONE IN THE MIDDLE OF EACH SPAN. PRISMATIC BEAM. Fig. 275. Let each half-span $= \frac{1}{2}l$. Neglect the weight of the beam. Required the reactions of the three supports. Call them P_B, P_O, and P_C. From symmetry $P_B = P_C$, and the tangent to the elastic curve at O is horizontal; and since the supports are <u>on a level</u> the deflection of C (and B) below O's tangent is zero. The <u>separate</u> elastic curves OD and DC have a common slope and a common ordinate at D.

For the EQUATION OF OD, make a section n anywhere between O and D, considering nC a free body, Fig. 276, with origin and axes as there indicated. From Σ(moms. about neutral axis of n) $= 0$ we have (see § 231)

$$EI\frac{d^2y}{dx^2} = P\left(\frac{l}{2} - x\right) - P_c(l - x) \quad \cdots\cdots (1)$$

$$\therefore \quad EI\frac{dy}{dx} = P\left(\frac{l}{2}x - \frac{x^2}{2}\right) - P_c\left(lx - \frac{x^2}{2}\right) + (C = 0) \cdots (2)$$

The constant $= 0$ for at O both x, and $dy \div dx$, $= 0$.

Taking the x-anti-derivative of (2) we have

$$EIy = P\left(\frac{lx^2}{4} - \frac{x^3}{6}\right) - P_c\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] \cdots\cdots (3)$$

Here again the constant is zero since at O x and y both $= 0$. (3) is the equation of OD, and allows no value of $x < 0$ or $> \frac{l}{2}$. It contains the unknown force P_c.

For the EQUATION OF DC, let the variable section n be made anywhere between D and C and we have (Fig. 276a; x may now range between $\frac{l}{2}$ and l)

$$EI\frac{d^2y}{dx^2} = -P_c(l-x) \quad \text{- - - - - - - - - -} \quad (4)$$

$$\therefore \quad EI\frac{dy}{dx} = -P_c\left(lx - \frac{x^2}{2}\right) + C' \qquad (5)'$$

To determine C', put $x = \frac{l}{2}$ both in $(5)'$ and (2), and equate the results (for the two curves have a common tangent line at D) whence $C' = \frac{1}{8}Pl^2$

$$\therefore \quad EI\frac{dy}{dx} = \frac{1}{8}Pl^2 - P_c\left(lx - \frac{x^2}{2}\right) \quad \text{- - - - - - -} \quad (5)$$

Hence $$EIy = \frac{1}{8}Pl^2x - P_c\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] + C'' \quad \text{- - - -} \quad (6)'$$

At D the curves have the same y, hence put $x = \frac{l}{2}$ in the right hand member both of (3) and $(6)'$, equating results, and we derive $C'' = -\frac{1}{48}Pl^3$

$$\therefore \quad EIy = \frac{1}{8}Pl^2x - P_c\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] - \frac{1}{48}Pl^3 \quad \text{- - - -} \quad (6)$$

which is the equation of DC, but contains the unknown reaction P_c. To determine P_c we employ the fact that O's tangent passes through C, (supports on <u>same</u> <u>level</u>) and hence when $x = l$ in (6), y is known to be zero. Making these substitutions in (6) we have

$$0 = \frac{1}{8}Pl^3 - \frac{1}{3}P_cl^3 - \frac{1}{48}Pl^3 \quad \therefore \; P_c = \frac{5}{16}P$$

From symmetry P_B also $= \frac{5}{16}P$, while P_0 must $= \frac{22}{16}P$, since $P_B + P_0 + P_c = 2P$, (whole beam free) [<u>Note</u>. If the supports were not on a level, but if (for instance) the middle support O were a small distance $= h_0$ below the level line joining the others, we should put $x = l$ and $y = -h_0$ in eq. (6), and thus obtain $P_B = P_c = \frac{5}{16}P + 3EI\frac{h_0}{l^3}$, which depends on the material and form of the prismatic beam and upon the length of one span, (whereas with supports <u>all on a level</u>, $P_B = P_c = \frac{5}{16}P$ is independent of the material and form of the beam so long as it is homogeneous and prismatic.) If P_0, which would then $= \frac{22}{16}P - 6EI(h_0 \div l^3)$, is found to be <u>negative</u>, it shows that O requires a support from above, in-

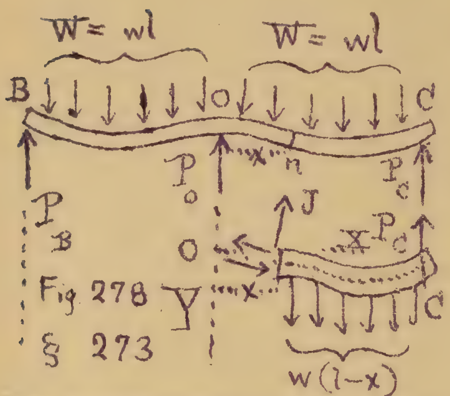


Fig 278 § 273

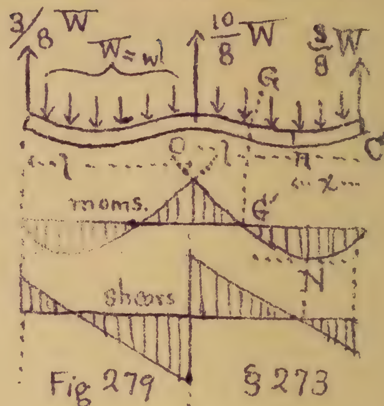


Fig 279 § 273

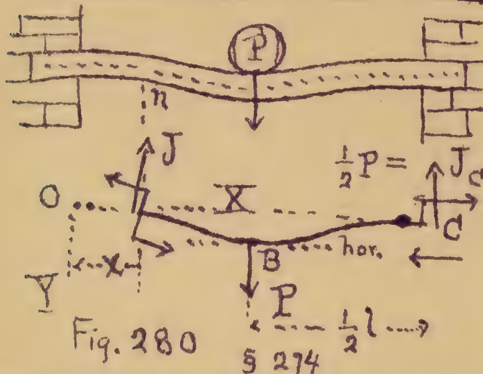


Fig. 280 § 274

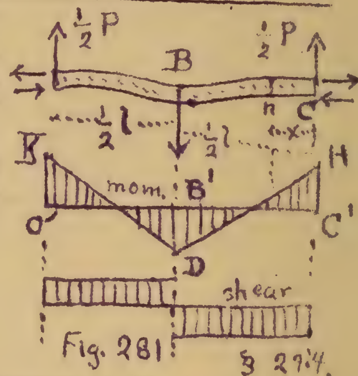


Fig. 281 § 274

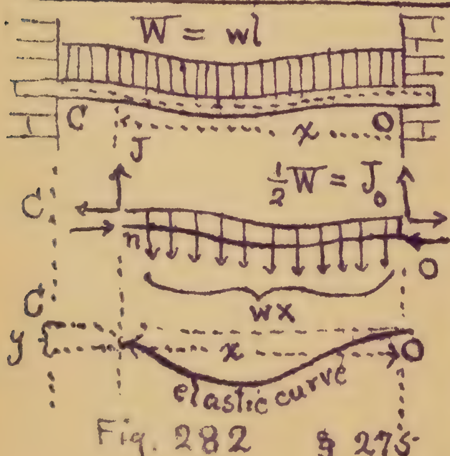


Fig. 282 § 275

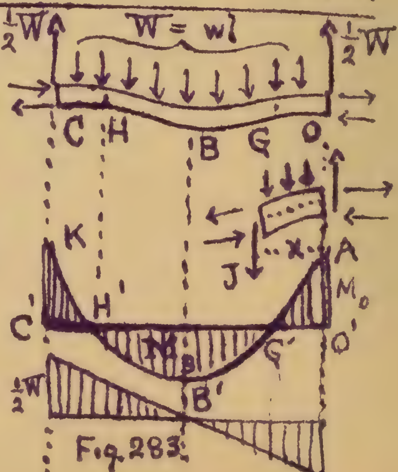


Fig. 283

stead of below, to cause it to occupy a position h_0 below the other supports; i.e. the beam must be "latched down" at O.]

The moment diagram of this case can now be easily constructed; Fig. 277. For any free body nC, n lying in DC, we have

$$M = \frac{5}{16} Px \ ,$$

i.e. varies directly as x, until x passes D when, for any point on DO,

$$M = \frac{5}{16} Px - P\left(x - \frac{l}{2}\right)$$

which is $=0$, (point of inflection of elastic curve) for $x = \frac{8}{11} l$ (note that x is measured from C in this figure) and at O, where $x = l$, becomes $-\frac{6}{32} Pl$

$$\therefore \quad M_0 = -\frac{6}{32} Pl\ ; \ M_Q = 0\ ; \ M_D = \frac{5}{32} Pl\ ; \text{and } M_C = 0$$

Hence, since M max. $= \frac{6}{32} Pl$, the equation for safe loading is

$$\frac{RI}{e} = \frac{6}{32} Pl \quad \text{------} \quad (7)$$

The shear at C and anywhere on $CD = \frac{5}{16} P$, while on DO it $= \frac{11}{16} P$ in the opposite direction

$$\therefore \quad J_m = \frac{11}{16} P \quad \cdots\cdots (8)$$

The moment and shear diagrams are easily constructed, as shown in Fig. 277, the former being symmetrical about a vertical line through O, the latter about the point O". Both are bounded by right lines.

273. TWO EQUAL SPANS. UNIFORMLY DISTRIBUTED LOAD OVER WHOLE LENGTH. PRISMATIC BEAM. Fig. 278. Supports B, O, C, on a level. Total load $= 2W = 2wl$ and may include that of the beam; w is constant. As before from symmetry $P_B = P_C$, the unknown reactions at the extremities.

Let $On = x$; then with nC free, Σ moms. about $n = 0$ gives

$$EI \frac{d^2y}{dx^2} = w(l-x)\left(\frac{l-x}{2}\right) - P_C(l-x) = \frac{w}{2}\left[l^2 - 2lx + x^2\right] - P_C(l-x) \cdots (1)$$

$$\therefore EI\frac{dy}{dx} = \frac{w}{2}\left[l^2x - lx^2 + \frac{x^3}{3}\right] - P_c\left[lx - \frac{x^2}{2}\right] + [\text{Const.} = 0] \cdots (2)$$

[Const. = 0 for at O both $dy \div dx$ the slope, and x, are $= 0$]

$$\therefore EIy = \frac{w}{2}\left[\frac{1}{2}l^2x^2 - \frac{1}{3}lx^3 + \frac{1}{12}x^4\right] - P_c\left[\frac{1}{2}lx^2 - \frac{1}{6}x^3\right] + (C = 0) \cdots (3)$$

[Const. = 0 for at O both x and y are $= 0$] Equations (1), (2), and (3) admit of any value of x from 0 to l, i.e. hold good for any point of the elastic curve OC, the loading on which follows a continuous law (viz.: w = constant) But when $x = l$, i.e. at C, y is known to be equal to zero, since O, B and C are on the axis of X, (tangent at O). With these values of x and y in eq. (3) we have

$$0 = \frac{w}{2}\cdot\frac{l^4}{4} - \frac{1}{3}P_c l^3 \quad \therefore P_c = \frac{3}{8}wl = \frac{3}{8}W$$

$$\therefore \quad P_B = \frac{3}{8}W \quad \text{and} \quad P_o = 2W - 2P_c = \frac{10}{8}W$$

The MOMENT AND SHEAR DIAGRAMS can now be formed since all the external forces are known. In Fig. 279 measure x from C. Then in any section n the moment of the "stress-couple" is

$$M = \frac{3}{8}Wx - \frac{wx^2}{2} \quad \text{----} \quad (1)$$

which holds good for any value of x on CO, i.e. from $x = 0$ up to $x = l$. By inspection it is seen that for $x = 0$, $M = 0$; and also for $x = \frac{3}{4}l$, $M = 0$, at the <u>inflection point</u> G, beyond which, toward O, the upper fibres are in tension the lower in compression, whereas between C and G they are vice versâ. For $x = l$, we have the moment at O, $M_o = -\frac{1}{8}wl^2 = \frac{1}{8}Wl$. As to the greatest moment to be found on CG, put $dM \div dx = 0$ and solve for x. This gives

$$\frac{3}{8}W - wx = 0 \quad \therefore [x \text{ for } M\,\text{max.}] = \frac{3}{8}l$$

which in eq. (1) gives

$$M_{\text{II}}\ (\text{at } N, \text{see figure}) = +\frac{9}{128}Wl$$

But this is numerically less than M_0 $(=-\frac{1}{8}Wl)$ hence the equation for safe loading is

$$\frac{R'I}{e}=\frac{1}{8}Wl \quad ------ \quad (7)$$

the same as if the beam were cut through at O, each half, of length l, retaining the same load as before [see §242 eq. (2)] Hence making the girder continuous over the middle support does not make it any stronger under a uniformly distributed load; but it does make it considerably stiffer.

As for the shear, J, we obtain it for any section by taking the x-derivative of M in eq. (1), or by putting Σ (vertical forces) $=0$ for the free body nC, and thus have for any section on CO

$$J=\frac{3}{8}W-wx$$

J is zero for $x=\frac{3}{8}l$ (where M reaches its calculus maximum M_N; see above) and for $x=l$ it $=-\frac{5}{8}W$ which is numerically greater than $\frac{3}{8}W$ its value at C. Hence

$$J_m=\frac{5}{8}W \quad --------- \quad (8)$$

The moment curve is a parabola (a separate one for each span), the shear curve a straight line, inclined to the horizontal, for each span.

PROBLEM. How would the reactions in fig. 278 be changed if the support O were lowered a (small) distance h_0 below the level of the other two?

274. PRISMATIC BEAM FIXED HORIZONTALLY AT BOTH ENDS (AT SAME LEVEL). SINGLE LOAD AT MIDDLE. Fig. 280 [As usual the beam is understood to be homogeneous so that E is the same at all sections] The building in, or fixing, of the two ends is supposed to be of such a nature as to cause no horizontal constraint; i.e., the beam does not act as a cord or chain, in any manner, and hence the sum of the horizontal components of the stresses in any section is zero, as in all

preceding cases of flexure. In other words the neutral axis still contains the centre of gravity of the section and the tensions and compressions are equivalent to a couple (the stress-couple) whose moment is the "moment of flexure".

If the beam is conceived cut through close to both wall-faces, and this portion, of length $= l$, considered free, the forces holding it in equilibrium consist of the downward force P (the load); two upward shears J_0 and J_c (one at each section); and two "stress-couples", one in each section, whose moments are M_0 and M_c. From symmetry we know that $J_0 = J_c$, and that $M_0 = M_c$. From $\Sigma Y = 0$ for the free body just mentioned, (but not shown in the figure), and from symmetry, we have $J_0 = \frac{1}{2}P$ and $J_c = \frac{1}{2}P$; but to determine M_0 and M_c, the form of the elastic curves OB and BC must be taken into account, as follows:

EQUATION OF OB. Fig. 280. Σ[mom. about neutral axis of any section n on OB] $=0$ (for the free body nC which has a section exposed at each end, n being the variable section) will give

$$EI\frac{d^2y}{dx^2} = P(\tfrac{1}{2}l - x) + M_c - \tfrac{1}{2}P(l-x) \quad \text{-----} \quad (1)$$

[NOTE. In forming this moment equation, notice that M_c is the sum of the moments of the tensions and compressions at C about the neutral axis at n, just as much as about the neutral axis of C; for those tensions and compressions are equivalent to a couple, and hence the sum of their moments is the same taken about any axis whatever ⊥ to the plane of the couple (§ 32).]

Taking the x-anti-derivative of each member of (1),

$$EI\frac{dy}{dx} = P(\tfrac{1}{2}lx - \tfrac{1}{2}x^2) + M_c x - \tfrac{1}{2}P(lx - \tfrac{1}{2}x^2) \quad \text{-----} \quad (2)$$

The constant is not added as it is zero. Now from symmetry we know that the tangent-line to the curve OB at B is horizontal, i.e., for $x = \frac{l}{2}$, $dy \div dx = 0$, and these values in eq.

(2) give us

$$O = \frac{1}{8}Pl^2 + \frac{1}{2}M_c l - \frac{3}{16}Pl^2 \text{; whence } M_c = M_o = \frac{1}{8}Pl \quad \text{- - - (3)}$$

SAFE LOADING. Fig. 281. Having now all the forces which act as external forces in straining the beam OC, we are ready to draw the moment diagram and find M_m. For convenience measure x from C. For the free body nC, we have [see eq. (3)]

$$\frac{1}{2}Px - M_c + \frac{pI}{e} = 0 \quad \therefore M = \frac{1}{8}Pl - \frac{1}{2}Px \quad \text{- - - - - (4)}$$

Eq. (4) holds good for any section on CB. By putting x = 0 we have $M = M_c = \frac{1}{8}Pl$; lay off $HC' = M_c$ to scale (so many inch-pounds moment to the inch of paper). At B, for $x = \frac{1}{2}l$, $M_B = -\frac{1}{8}Pl$; hence lay off $B'D = \frac{1}{8}Pl$ on the opposite side of the axis O'C' from HC', and join DH. DK, symmetrical with DH about B'D, completes the moment curves, viz.: two right lines. The max. M. is evidently $= \frac{1}{8}Pl$ and the equation of safe loading

$$\frac{R'I}{e} = \frac{1}{8}Pl \quad \text{- - - - - - - - (5)}$$

Hence the beam is twice as strong as if simply supported at the ends, under this load; it may also be proved to be four times as stiff.

The points of inflection of the elastic curve are in the middles of the half-spans, while the max. shear is

$$J_m = \frac{1}{2}P \quad \text{- - - - - - - - (6)}$$

275. PRISMATIC BEAM FIXED HORIZONTALLY AT BOTH ENDS [AT SAME LEVEL]. UNIFORMLY DISTRIBUTED LOAD OVER THE WHOLE LENGTH. Fig. 282. As in the preceding problem, we know from symmetry that $J_o = J_c = \frac{1}{2}W = \frac{1}{2}wl$, and that $M_o = M_c$, and determine the latter quantities by the equation of the curve OC, there being but one curve in the present instance, instead of two, as there is no change in the law of loading between O and C. With nO free, $\Sigma(\text{mom}_n) = 0$ gives

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}Wx - M_o - \frac{wx^2}{2} \quad \text{- - - - - - (1)}$$

and ∴ $$EI\frac{dy}{dx} = \frac{1}{2}W\frac{x^2}{2} - M_0 x - \frac{wx^3}{6} + [C=0] \quad \cdots\cdots (2)$$

The tangent line at O being horizontal we have for $x=0$, $\frac{dy}{dx}=0$, ∴ $C=0$. But since the tangent line at C is also horizontal, we may for $x=l$ put $dy \div dx = 0$, and obtain

$$0 = \frac{1}{4}Wl^2 - M_0 l - \frac{1}{6}wl^3; \text{ whence } M_0 = \frac{1}{12}Wl \quad --- (3)$$

as the moment of the stress-couple close to the wall at O, and at C.

Hence, Fig. 283, the equation of the moment curve (a single continuous curve in this case) is found by putting $\Sigma(\text{mom}_n)=0$ for the free body nO of length x, thus obtaining

$$\frac{pI}{e} + \frac{1}{2}Wx - M_0 - \frac{wx^2}{2} = 0$$

i.e. $$M = \frac{1}{12}Wl + \frac{wx^2}{2} - \frac{1}{2}Wx \quad ---- (4)$$

an equation of the second degree, indicating a conic. At O $M = M_0$ of course, $= \frac{1}{2}Wl$; at B by putting $x = \frac{1}{2}l$ in (4), we have $M_B = -\frac{1}{24}Wl$, which is less than M_0, although M_B is the calculus max. (negative) for M, as may be shown by writing the expression for the shear $(J = \frac{1}{2}W - wx)$ equal to zero, etc.

Hence $M_m = \frac{1}{12}Wl$ and the equation for <u>safe loading</u> is

$$\frac{R'I}{e} = \frac{1}{12}Wl \quad -------- (5)$$

Since (with this form of loading) if the beam were not built in but simply rested on two end supports the equation for safe loading would be $[R'I \div e] = \frac{1}{8}Wl$ (see §242), it is evident that with the present mode of support it is 50 per cent. stronger as compared with the other; i.e. as regards normal stresses in the outer elements. As regards shearing stresses in the web if it has one, it is no stronger, since $J_m = \frac{1}{2}W$ in both cases.

As to <u>stiffness</u> under the uniform load, the max. deflection in the present case may be shown to be only $\frac{1}{5}$ of that in the case of the simple end supports.

It is noteworthy that the shear diagram in Fig. 283 is iden-

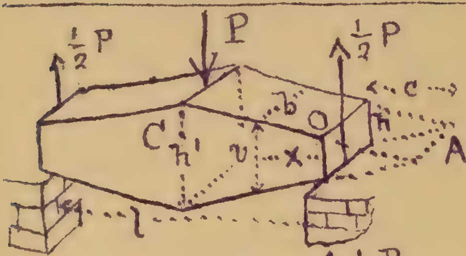
ical with that for simple end supports §242, under uniform load; while the moment diagrams differ as follows: The parabola KBA, Fig. 283, is identical with that in Fig. 235, but the horizontal axis from which the ordinates of the former are measured, instead of joining the extremities of the curve, cuts it in such a way as to have equal areas between it and the curve, on opposite sides;

i.e. areas [KC'H' + AG'O'] = area HGB'

In other words, the effect of fixing the ends horizontally is to shift the moment parabola upward a distance = M_c (to scale), = $\frac{1}{12}Wl$, with regard to its axis of reference.

276. REMARKS. The foregoing very simple cases of continuous girders illustrate the means employed for determining the reactions of supports and eventually the max. moment and the equations for safe loading and for deflections. When there are more than three supports, with spans of unequal length, and loading of any description, the analysis leading to the above results is much more complicated and tedious, but is considerably simplified and systematized by the use of the remarkable THEOREM OF THREE MOMENTS, the discovery of Clapeyron, in 1857. By this theorem, given the spans, the loading, and the relative vertical heights of the supports, we are enabled to write out a relation between the moments of each three consecutive supports, and thus obtain a sufficient number of equations to determine the moments at all the supports [p. 641 Rankine's Applied Mechanics.] From these moments the shears close to each side of each support are found, then the reactions, and from these and the given loads the moment at any section can be determined; and hence finally the max. moment M_m, and the max. shear J_m.

This theorem and its application are comparatively simple by graphic methods, and their presentation is therefore deferred.



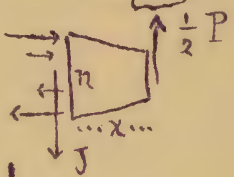


Fig. 284

§ 278

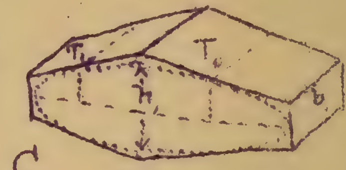
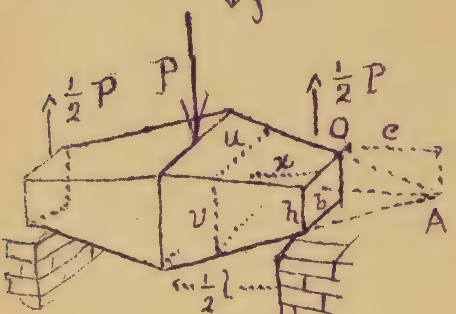


Fig. 287 § 281



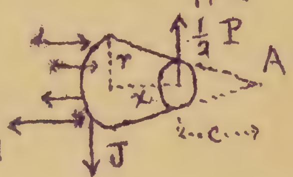


Fig. 286

§ 281

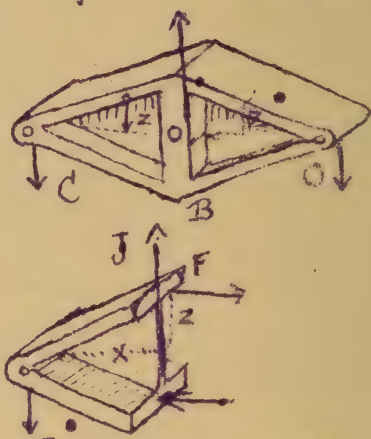
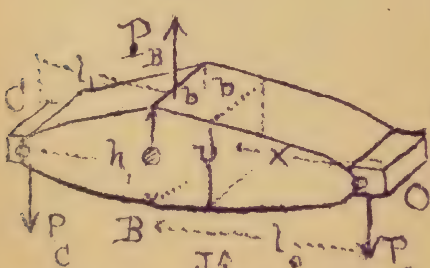


Fig. 288 § 282



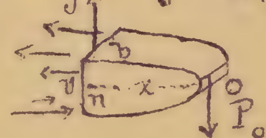


Fig. 286

§ 281

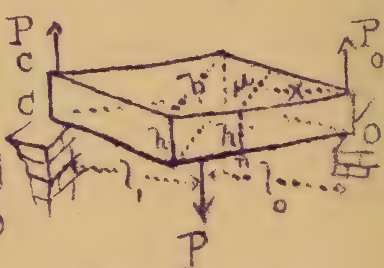


Fig. 289

§ 283.

THE DANGEROUS SECTION IN NON-PRISMATIC BEAMS.

277. REMARKS. By "dangerous section" is meant that section (in a given beam under given loading with given mode of support) where the normal stress in the outer fibre, at distance e from its neutral axis, is greater than in the outer fibre of any other section. Hence the elasticity of the material will first be impaired in the outer fibre of this section, if the load is gradually increased in amount (but not altered in distribution).

In all preceding problems, the beam being prismatic, I, the moment of inertia, and e were the same in all sections, hence when the equation $\frac{pI}{e} = M$ [§239] was solved for p, giving

$$p = \frac{Me}{I} \qquad (1)$$

we found that p was a max., $= p_m$, for that section whose M was a maximum, since p varied as M, the moment of the stress-couple, as successive sections along the beam were examined.

But for a non-prismatic beam I and e change, from section to section, as well as M, and the ordinate of the moment diagram no longer shows the variation of p, nor is p a max. where M is a max. To find the dangerous section, then, for a non-prismatic beam, we express the M, the I, and the e of any section in terms of x, thus obtaining $p = \text{func.}(x)$, then writing $dp \div dx = 0$, and solving for x.

278. DANGEROUS SECTION IN A DOUBLE TRUNCATED WEDGE. TWO END SUPPORTS. SINGLE LOAD IN MIDDLE. The form is shown in Fig. 284. Neglect weight of beam; measure x from one support, O. The reaction at each support is $\frac{1}{2}P$. The width of the beam $= b$ at all sections, while its height, v, varies, being $= h$ at O. To express the $e = \frac{1}{2}v$, and the $I = \frac{1}{12}bv^3$ (§247) of any section on OC in terms of x, conceive the sloping faces of the truncated wedge to be prolonged to their intersection

A, at a known distance $=c$ from the face at O. We then have from similar triangles

$$v : x+c :: h : c \quad , \therefore \; v = \frac{h}{c}(x+c) \quad \cdots\cdots \quad (1)$$

$$\text{and } \therefore \; e = \frac{1}{2}\frac{h}{c}(x+c) \text{ and } I = \frac{1}{12} b \frac{h^3}{c^3}[x+c]^3 \quad \cdots\cdots (2)$$

For the free body nO, $\Sigma(\text{moms}_n)=0$ gives

$$\frac{1}{2}Px - \frac{pI}{e} = 0 \quad \therefore \; p = \frac{Pxe}{2I} \quad \cdots\cdots \quad (3)$$

[That is, the $M = \frac{1}{2}Px$]. But from (2), (3) becomes

$$p = 3P\frac{c^2}{h^2}\cdot\frac{x}{(x+c)^2}; \text{ and } \frac{dp}{dx} = 3P\frac{c^2}{h^2}\cdot\frac{(x+c)^2 - 2x(x+c)}{(x+c)^4} \quad (4)$$

By putting $dp \div dx = 0$ we obtain both $x=-c$, and $x=+c$, of which the latter, $x=+c$, corresponds to a maximum for p (since it will be found to give a negative result on substitution in $d^2p \div dx^2$).

Hence the dangerous section is as far from the support O, as the imaginary edge, A, of the completed wedge, but of course on the opposite side. This supposes that the half-span, $\frac{1}{2}l$, is $> c$; if not, the dangerous section will be at C', the middle of the beam, as if the beam were prismatic.

Hence, with $\frac{1}{2}l < c$ the equation for safe loading is: (h' = height at middle)

$$\frac{R'bh'^2}{6} = \frac{1}{4}Pl \quad --(5)$$

while with $\frac{1}{2}l > c$ the equation for safe loading is (put $x=c$ and $p=R'$ in (3))

$$\frac{R'b[2h]^2}{6} = \frac{1}{2}Pc \quad --(6)$$

279. DOUBLE TRUNCATED PYRAMID AND CONE. Fig. 285. For the truncated pyramid both width $=u$, and height $=v$ are variable, and if b and h are the dimensions at O, and $c = \overline{OA}$ = distance from O to the imaginary vertex A, we shall have from similar triangles $u = \frac{b}{c}(x+c)$ and $v = \frac{h}{c}(x+c)$. Hence, substituting $e = \frac{1}{2}v$ and $I = \frac{1}{12}uv^3$, in the moment equation

$$\frac{pI}{e} - \frac{1}{2}Px = 0 \quad \text{we have} \quad p = 3P\cdot\frac{bh^2}{c^3}\cdot\frac{x}{(x+c)^3} \quad \cdots (7)$$

$$\therefore \frac{dp}{dx} = 3P\frac{bh^2}{c^3}\cdot\frac{(x+c)^3 - 3x(x+c)^2}{(x+c)^6} \quad \ldots\ldots \quad (8)$$

Putting this $=0$, we have $x=-c$, $x=-c$, and $x=+\frac{1}{2}c$. hence the dangerous section is at a distance $x=\frac{1}{2}c$ from O, and the equation for safe loading is

$$\text{either } \frac{R'b'h'^2}{6} = \frac{1}{4}Pl \quad \text{if } \tfrac{1}{2}l \text{ is} < \tfrac{1}{2}c \quad \ldots \quad (9)$$

(in which b' and h' are the dimensions at mid-span)

$$\text{or } \frac{R\left(\frac{3}{2}b\right)\left(\frac{3}{2}h\right)^3}{6} = \frac{1}{4}Pc \quad \text{if } \tfrac{1}{2}l \text{ is} > \tfrac{1}{2}c \quad \ldots \quad (10)$$

For the TRUNCATED CONE where $e=$ the variable radius r, and $I=\frac{1}{4}\pi r^4$, we also have

$$p = [\text{Const.}]\cdot\frac{x}{(x+c)^3} \quad \ldots \quad (11)$$

and hence p is a max. for $x=\frac{1}{2}c$, and the equation for safe loading

$$\text{either } \frac{R'r'^3}{4} = \frac{1}{4}Pl, \text{ for } \tfrac{1}{2}l < \tfrac{1}{2}c \quad (12)$$

(where $r'=$ radius of mid-span section);

$$\text{or } \frac{R'\left(\frac{3}{2}r\right)^3}{4} = \frac{1}{4}Pc, \text{ for } \tfrac{1}{2}l > \tfrac{1}{2}c \quad \ldots \quad (13)$$

NON-PRISMATIC BEAMS OF "UNIFORM STRENGTH."

280. REMARKS. A beam is said to be of "uniform strength", when its form, its mode of support, and the distribution of loading, are such, that the normal stress p has the same value in all the outer fibres, and thus one element of economy is secured, viz.: that all the outer fibres may be made to do full duty, since under the safe loading, p will be $=R'$ in all of them. [Of course, in all cases of flexure the elements between the neutral surface and the outer fibres being under tensions and compressions less than R' per sq. inch are not doing full duty, as regards economy of material, unless perhaps with respect to shearing stresses] In fig. 265, §261, we have already had an instance of a body of uniform strength

in flexure, viz.: the middle segment, CD, of that figure; for the moment is the same for all sections of CD [eq. (2) of that §] and hence the normal stress p in the outer fibres (the beam being prismatic in that instance)

In the following problems the weight of the beam itself is neglected. The general method pursued will be to find an expression for the outer-fibre-stress p, at a definite section of the beam, where the dimensions of the section are known or assumed, then an expression for p in the variable section, and equate the two.

281. PARABOLIC WORKING BEAM. UNSYMMETRICAL. Fig. 286. CBO is a working beam or lever, B being the fixed fulcrum or bearing. The force P_0 being given we may compute P_c from the mom. equation $P_0 l_0 = P_c l_1$, while the fulcrum reaction is $P_B = P_0 + P_c$. All the forces are ⊥ to the beam. The beam is to have the same width b at all points, and is to be rectangular in section.

Required FIRST, the proper height h_1 at B, for safety. From the free body BO, of length $= l_0$, we have $\Sigma(\text{moms}_B) = 0$; i.e.,

$$\frac{p_B I}{e} = P_0 l_0 \;;\; \text{or}\; p_B = \frac{6 P_0 l_0}{b h_1^2} \quad \ldots\ldots\ldots (1)$$

Hence, putting $p_B = R'$, h' becomes known from (1).

Required, SECONDLY, the relation between the variable height v (at any section n) and the distance x of n from O. For the free body nO, we have $(\Sigma \text{ moms}_n = 0)$

$$\frac{p_n I}{e} = P_0 x \;;\; \text{or}\; \frac{p_n \frac{1}{12} b v^3}{\frac{1}{2} v} = P_0 x \;\text{ and }\therefore\; p_n = \frac{6 P_0 x}{b v^2} \quad \ldots (2)$$

But for "uniform strength" p_n must $= p_B$; hence equate their values from (1) and (2) and we have

$$\frac{x}{v^2} = \frac{l_0}{h_1^2}\text{, which may be written } \left(\tfrac{1}{2}v\right)^2 = \frac{\left(\frac{1}{2}h_1\right)^2}{l_0}\, x \quad \ldots (3)$$

so as to make the relation between the abscissa x and the ordinate $\frac{1}{2}v$ more marked; it is that of a parabola, whose

vertex is at O.

The parabolic outline for the portion BC is found similarly. The local stresses at C, B, and O must be properly provided for by evident means. The shear $J = P_0$, at O, also requires special attention.

This shape of beam is often adopted in practice for the working beams of engines, etc.

The parabolic outlines just found may be replaced by trapezoidal forms, fig. 287, without using much more material, and by making the sloping plane faces tangent to the parabolic outline at points T_0 and T_1, half-way between O and B, and C and B, respectively, it can be proved that they contain minimum volumes, among all trapezoidal forms capable of circumscribing the given parabolic bodies. The dangerous sections of these trapezoidal bodies are at the tangent points T_0 and T_1. This is as it should be, (see § 278), remembering that the subtangent of a parabola is bisected by the vertex.

282. I-BEAM OF UNIFORM STRENGTH. Support and load same as in the preceding §. Fig. 288. Let the area of the flange-section be $= F$ and let it be the same for all values of x. Considering all points of F at any one section as at the same distance z from the neutral axis, we may write $I = z^2 F$, and assuming that the flanges take all the tension and compression while the (thin) web carries the shear, the free body of length x in fig. 288 gives (moms. about n)

$$\frac{pI}{e} = P_c x \text{; i.e. } \frac{pz^2F}{z} = P_c x \text{ : or, since } p \text{ is to be constant,}$$

$$z = [\text{Const}].x \quad \text{- - - - - - - - - - - -} \quad (1)$$

i.e. z must be made proportional to x.

Hence the flanges should be made straight. Practically, since they unite at C, the web takes but little shear.

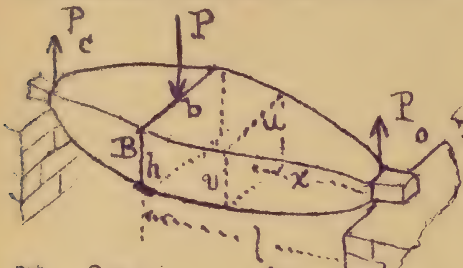


Fig. 290 § 284

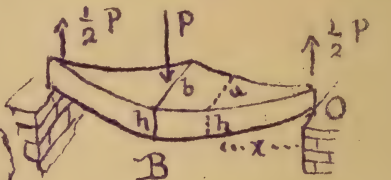


Fig. 291a § 290

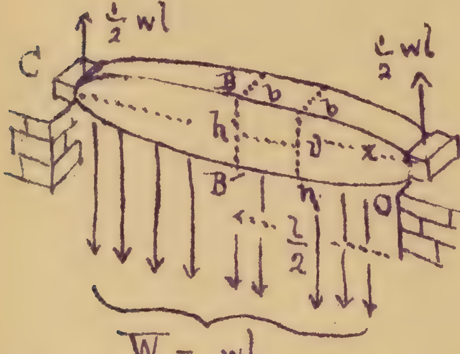


Fig. 291 § 286

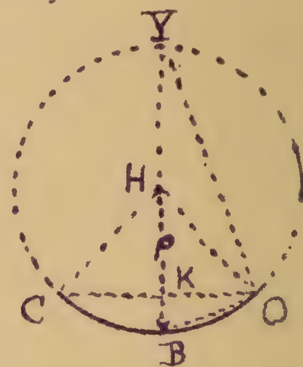


Fig. 292 § 290

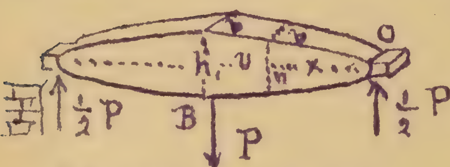


Fig. 293 § 291

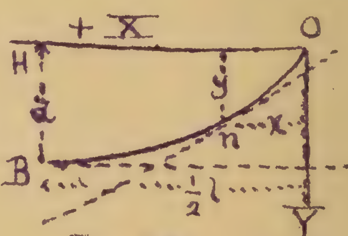
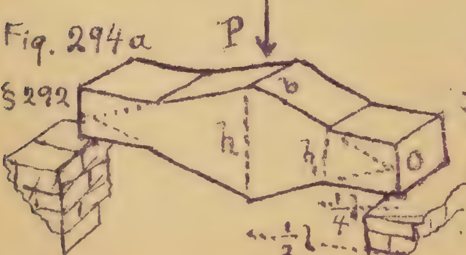


Fig. 294 § 291

Fig. 294a § 292



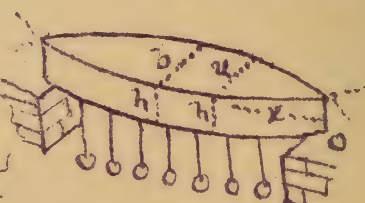


Fig. 295 § 293

283. RECTANG. SECTION. HEIGHT CONSTANT. TWO SUPPORTS (AT EXTREMITIES). SINGLE ECCENTRIC LOAD. Fig. 289. b and h are the dimensions of the section at B. With BO free we have

$$\frac{p_B I_B}{e_B} - P_0 l_0 = 0 \quad \therefore p_B = \frac{6 P_0 l_0}{b h^2} \quad \ldots\ldots (1)$$

At any other section on BO, as n, where the width is u, the variable whose relation to x is required; we have for nO free

$$\frac{p_n I_n}{e_n} = P_0 x\,;\ \text{or}\ \frac{p_n \frac{1}{12} u h^3}{\frac{1}{2} h} = P_0 x \quad \therefore p_n = \frac{6 P_0 x}{u h^2} \ldots (2)$$

Equating p_B and p_n we have $u : b :: x : l_0$ .... (3)

That is, BO must be wedge shaped with its edge at O, vertical.

284. SIMILAR RECTANG. SECTIONS. OTHERWISE AS BEFORE. Fig 290. b and h are the dimensions at B; at any other section n, on BO, the height v and width u, are the variables whose relation to is desired and by hypothesis are connected by the relation $u : v :: b : h$ ...... (1)

(since the section at n is a rectangle similar to that at B.)

For the free body BO.... $p_B = \dfrac{6 P_0 l_0}{b h^2}$ ...... (2)

For the free body nO.... $p_n = \dfrac{6 P_0 x}{u v^2}$ .... (3)

Writing $p_n = p_B$ we obtain $l_0 \div b h^2 = x \div u v^2$, in which put $u = bv \div h$, from (1); whence

$$\frac{v^3}{h^3} = \frac{x}{l}\,;\ \text{or}\ \left(\tfrac{1}{2} v\right)^3 = \left(\tfrac{1}{2} h\right)^3 \frac{x}{l} \quad \ldots\ldots (4)$$

which is the equation of the curve (a cubic parabola) whose abscissa is x and ordinate $\frac{1}{2} v$; i.e., of the upper curve of the outline of the central longitudinal vertical plane section of the body (dotted line BO) which is supposed symmetrical

about such a plane. Similarly the central horizontal plane section will cut out a curve a quarter of which (dotted line B'O) has an equation $(\frac{1}{2}u)^3 = (\frac{1}{2}b)^3 \frac{x}{l}$ . . . . . (5)

That is, the height and width must vary as the cube root of the distance from the support. The portion CB will give corresponding results, referred to the support C.

[If the beam in this problem is to have circular cross-sections, let the student treat it in the same manner.]

286. UNIFORM LOAD. TWO END SUPPORTS. RECTANGULAR CR. SECTIONS. WIDTH CONSTANT. How should the height vary, the height and width at the middle being h and b? Fig. 291. From symmetry each reaction $= \frac{1}{2}W = \frac{1}{2}wl$. At any cross section n, the width is $= b$, (same as that at the middle) and the height $= v$, variable.

$\Sigma(\text{moms.}_n) = 0$, for the free body nO, gives

$$\frac{p_n I_n}{e_n} = \frac{1}{2}wlx - \frac{wx^2}{2}; \text{ i.e. } \frac{p_n \frac{1}{12} b v^3}{\frac{1}{2}v} = \frac{1}{2}wlx - \frac{wx^2}{2} \quad \ldots\ldots (1)$$

But to be a beam of uniform strength, p_n is to be $= p_B$ as computed from $\Sigma(\text{moms.}_B) = 0$ for the free body .. BO, i.e. from

$$\frac{p_B \frac{1}{12} b h^3}{\frac{1}{2}h} = \frac{1}{2}wl \cdot \frac{l}{2} - \frac{w(\frac{1}{2}l)^2}{2} \quad \ldots\ldots (2)$$

Hence solve (1) for p_n and (2) for p_B and equate the results, whence

$$v^2 = \frac{h^2}{(\frac{1}{2}l)^2}\left[lx - x^2\right]; \quad \text{or} \left(\frac{1}{2}v\right)^2 = \frac{(\frac{1}{2}h)^2}{(\frac{1}{2}l)^2}\left[lx - x^2\right] \ldots (3)$$

This relation between the abscissa x and the ordinate $\frac{1}{2}v$, of the curve CBO, shows it to be an ellipse since eq. (3) is that of an ellipse referred to its principal diameter and the tangent at its vertex as co-ordinate axes.

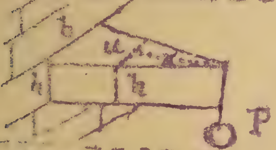
In this case eq. (3) covers the whole extent of both upper and lower curves, i.e. the complete outline, of the curve CBOB';

whereas in Figs. 286, 289, and 290, such is not the case.

287. CANTILEVERS of UNIFORM STRENGTH. Beams built in at one end, horizontally, and projecting from the wall without support at the other, should have the forms given below, for the given cases of loading, if all cross sections are to be RECTANGULAR and the weight of beam neglected. Sides of sections horizontal and vertical. Also, the sections are symmetrical about the axis of the piece. b and h are the dimensions at the wall. No proofs given.

Width constant, Vertical outline parabolic, Single end load

$$\left(\tfrac{1}{2}v\right)^2 = \left(\tfrac{1}{2}h\right)^2 \frac{x}{l} \quad \ldots(1)$$



Height const. Single end load Horiz. outline Triangular.

$$\left(\tfrac{1}{2}u\right) = \left(\tfrac{1}{2}b\right) \frac{x}{l} \quad \ldots(2)$$

Constant ratio of height v to width u. Both outlines cubic parabolas.

$$\left(\tfrac{1}{2}v\right)^3 = \left(\tfrac{1}{2}h\right)^3 \frac{x}{l} \quad \ldots(3)$$

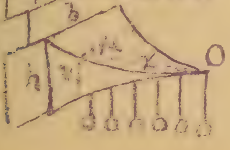
$$\left(\tfrac{1}{2}u\right)^3 = \left(\tfrac{1}{2}b\right)^3 \frac{x}{l} \quad \ldots(3)'$$

Uniform Load. Width constant. Vertical outline triangular

$$\left(\tfrac{1}{2}v\right) = \left(\tfrac{1}{2}h\right) \frac{x}{l} \quad \ldots(4)$$

Uniform Load. Height constant. Hor. outline = two parabolas meeting at O (vertex) with geomet. axes ∥ to wall

$$\tfrac{1}{2}u = \left(\tfrac{1}{2}b\right) \frac{x^2}{l^2} \quad \ldots(5)$$



Uniform Load. Both outlines semi-cubic parabolas

$$\left(\tfrac{1}{2}u\right)^2 = \left(\tfrac{1}{2}b\right)^2 \frac{x^3}{l^3} \quad \ldots(6)$$

$$\left(\tfrac{1}{2}v\right)^2 = \left(\tfrac{1}{2}h\right)^2 \frac{x^3}{l^3} \quad \ldots(6)'$$

289. Beams and cantilevers of circular cross-sections may be dealt with similarly, and the proper longitudinal outline given, to constitute them "bodies of uniform strength". As a consequence of the possession of this property, with loading and mode of support of specified character, the following may be stated; that to find the equation of safe loading any cross section whatever may be employed. This refers to tension and compression. As regards the shearing stresses in different parts of the beam the condition of "uniform strength" is not necessarily obtained at the same time with that for normal stress in the outer fibres.

DEFLECTION OF BEAMS OF UNIFORM STRENGTH.

290. CASE OF §283, the double wedge, but symmetrical, i.e. $l_1 = l_0 = \frac{1}{2}l$, Fig. 291. Here we shall find the use of the form $\frac{EI}{\rho}$, (of the three forms for the moment of the stress couple, see eqs. (5) (6) and (7) §§229 and 231) of the most direct service in determining the form of the elastic curve OB, which is symmetrical, and has a common tangent at B, with the curve BC. First to find the radius of curvature, ρ, at any section n, we have for the free body nO $\Sigma(\text{moms}_n = 0)$, whence

$$-\frac{EI}{\rho} + \frac{1}{2}Px = 0; \quad \text{but} \begin{Bmatrix}\text{from}\\ \text{eq. 3}\\ \S 283\end{Bmatrix} x = \frac{u}{b}\,\frac{1}{2}l \text{ and } I = \frac{1}{12}uh^3,$$

we have

$$\frac{1}{12}\,\frac{E}{\rho}\,uh^3 = \frac{1}{4}P\frac{ul}{b} \text{ and } \therefore \rho = \frac{1}{3}\,\frac{bh^3}{l}\cdot\frac{E}{P} \quad \text{--(1)}$$

from which all variables have disappeared in the left hand member; i.e. ρ is CONSTANT, the same at all points of the elastic curve, hence the latter is the arc of a circle, having a horizontal tangent at B.

To find the deflection, d, at B, consider Fig. 292

where $d = \overline{KB}$, and the full circle of radius $BH = \rho$ is drawn.

The triangle KOB is similar to YOB,
and ∴ $\overline{KB} : \overline{OB} :: \overline{OB} : \overline{YB}$

But $OB = \frac{1}{2}l$, $KB = d$ and $YB = 2\rho$

$$\therefore d = \frac{(\frac{1}{2}l)^2}{2\rho}, \text{ and } \therefore, \text{from eq. (1)}, d = \frac{3}{8} \cdot \frac{Pl^3}{bh^3E} \quad \cdots (2)$$

From eq. (4) § 233 we note that for a beam of the same material but prismatic (parallelopipedical in this case,) having the same dimensions, b and h, at all sections as at the middle, deflects an amount $= \frac{1}{48} \frac{Pl^3}{EI} = \frac{1}{4} \frac{Pl^3}{bh^3E}$ under a load P in the middle of the span. Hence the tapering beam of the present § has only $\frac{2}{3}$ the stiffness of the prismatic beam, for the same b, h, l, E, and P.

291. CASE OF § 281 (PARABOLIC BODY) WITH $l_1 = l_0$, i.e. SYMMETRICAL. Fig. 293. Required the equation of the neutral line OBC. For the free body nO $\Sigma(\text{moms.}_n) = 0$ gives us

$$EI \frac{d^2y}{dx^2} = -\frac{1}{2}Px \quad \cdots\cdots (1)''$$

Fig. 294 shows simply the geometrical relations of the problem, position of origin, axes, etc. OnB is the neutral line or elastic curve whose equation, and greatest ordinate d, are required. (The left hand member of eq. (1)'' is made negative because $d^2y \div dx^2$ is negative, the curve being concave to the axis X in this, the first quadrant.)

Now if the beam were prismatic, I, the "moment of inertia" of the cross section would be constant, i.e. the same for all values of x, and we might proceed by taking the x-anti-derivative of each member of (1)'' and add a constant, but it is variable and

$= \frac{1}{12} bu^3 = \frac{1}{12} \cdot \frac{bh_1^3}{(\frac{1}{2}l)^{3/2}} x^{3/2}$, (from eq. 3 §281 putting $l_0 = \frac{1}{2}l$) hence (1)' becomes

$$\frac{1}{12} E \frac{bh_1^3}{(\frac{1}{2}l)^{3/2}} x^{3/2} \frac{d^2y}{dx^2} = -\frac{1}{2} Px \quad \text{------} (1)'$$

To put this into the form Const. $\times \frac{d^2y}{dx^2}$ = func. of (x), we need only divide through by $x^{3/2}$, (and for brevity denote $\frac{1}{12} Ebh_1^3 \div (\frac{1}{2}l)^{3/2}$ by A) and obtain

$$A \frac{d^2y}{dx^2} = \frac{1}{2} Px^{-\frac{1}{2}} \quad \text{------} (1)$$

We can now take the x-anti-derivative of each member, and thus have

$$A \frac{dy}{dx} = -\frac{1}{2} P (2x^{-\frac{1}{2}}) + C \quad \text{---} (2)'$$

To determine the constant C, we utilize the fact that at B, where $x = \frac{1}{2}l$, the slope $dy \div dx$ is zero since the tangent line is there horizontal, whence from (2)'

$$0 = -P\sqrt{\frac{l}{2}} + C \quad \therefore C = P\sqrt{\frac{l}{2}}$$

$\therefore$ (2)' becomes $A \frac{dy}{dx} = P[\sqrt{\tfrac{1}{2}l} - x^{\frac{1}{2}}]$ ---- (2)

$$\therefore \quad Ay = P[\sqrt{\tfrac{1}{2}l} \cdot x - \tfrac{2}{3}x^{3/2}] + [C' = 0] \quad \text{---} (3)$$

($C' = 0$ since for $x = 0$, $y = 0$). We may now find the deflection d (Fig. 294) by writing $x = \frac{1}{2}l$ and $y = d$ whence, after restoring the value of the constant A,

$$d = \frac{1}{2} \frac{Pl^3}{Ebh_1^3} \quad \text{------} (4)$$

and is twice as great [being $= 2 \cdot \frac{Pl^3}{4Ebh_1^3}$] as if the girder were parallelopipedical. In other words the present girder is only half as stiff as the prismatic one.

292 SPECIAL PROBLEM. (I) The symmetrical beam in Fig. 294 is of rectangular cross section and

constant width $= b$, but the height is constant only over the extreme quarter spans being $= h_1 = \frac{1}{2} h$, i.e. half the height h at midspan. The convergence of the two truncated wedges forming the middle quarters of the beam is such that the prolongations of the upper and lower surfaces would meet over the supports, (as should be the case to make $h = 2h_1$). Neglecting the weight of the beam, and placing a single load in middle, it is required to find the equation for safe loading; also the equations of the elastic curves CB_1 and B_1B; and finally the deflection.

The solutions of this and the following problem are left to the student, as exercises. Of course the beam here given is not one of uniform strength.

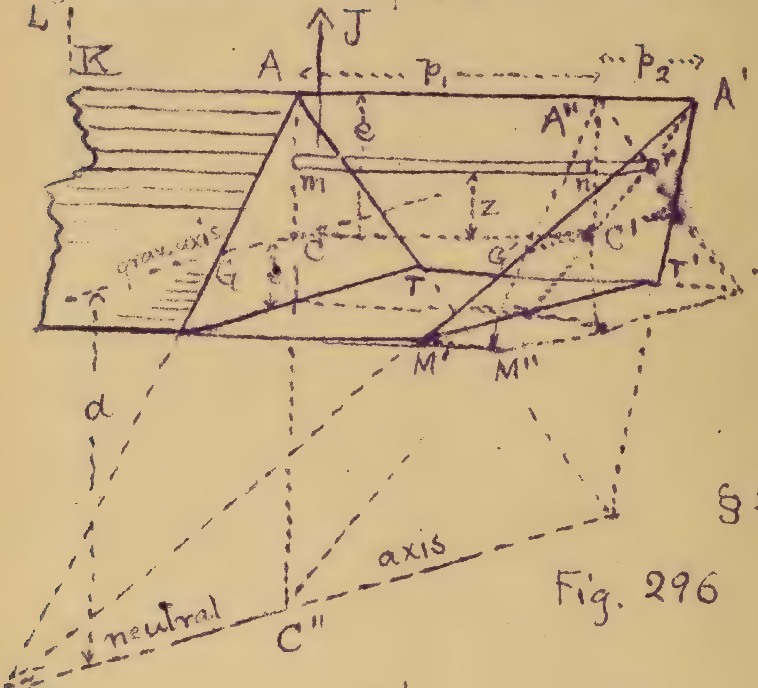
293. SPECIAL PROBLEM (II). Fig. 295. Required the manner in which the width of the beam must vary, the height being constant, cross-sections rectangular, weight of beam neglected, to be a beam of uniform strength, if the load is uniformly distributed?

CHAP. V. Flexure of Prismatic Beams under OBLIQUE FORCES.

294. REMARKS. By "oblique forces" will be understood external forces not perpendicular to the beam, but these external forces will be confined to one plane, called the force-plane, which contains the axis of the beam and also cuts the beam symmetrically. The curvature induced by these external forces will as before be considered very slight, so that distances measured along the beam will be treated as unchanged by the flexure.

It will be remembered that in previous problems the proof that the neutral axis of each cross section passes through its centre of gravity, rested on the fact that when a portion of the beam having a given section as one of its bounding surfaces is considered free the condition

Figs 296 to 295 To face p. 124 §§ 295



§ 295

Fig. 296

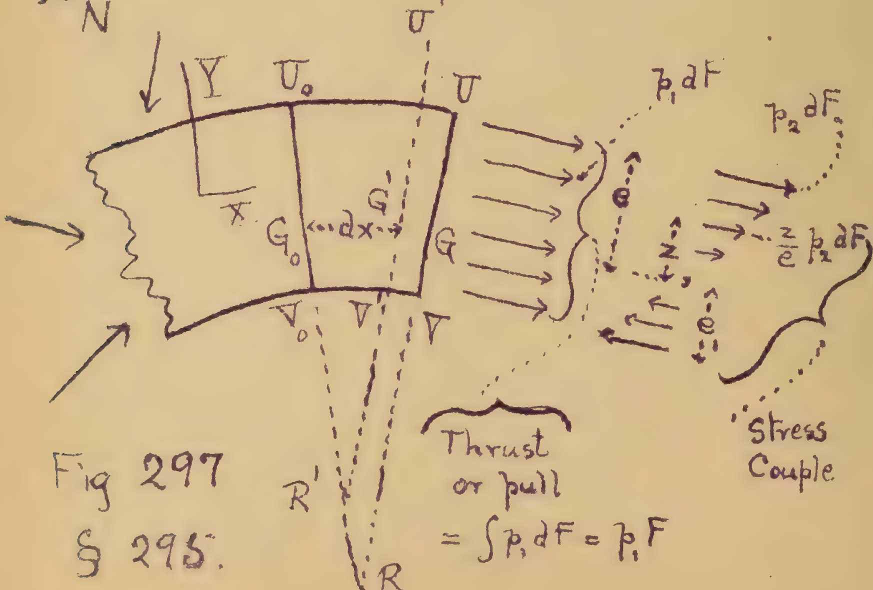


Fig 297

§ 295.

of equilibrium Σ (compons. ∥ to beam) = 0 does not introduce any of the external forces, since these in the problems referred to, were ⊥ to the beam; but in the problems of the present chapter such is not the case, and hence the neutral axis does not necessarily pass through the centre of gravity of any section, and in fact may have only an ideal, geometrical, existence, being entirely outside of the section; in other words the fibres whose ends are exposed in a given section may all be in tension, (or all in compression) of intensities varying with the distance of each from the neutral axis. It is much more convenient, however, to take for an axis of moments the gravity axis parallel to the neutral axis instead of the neutral axis itself, since this gravity axis has always a known position.

295. CLASSIFICATION OF THE ELASTIC FORCES. SHEAR, THRUST, AND STRESS-COUPLE. Fig. 296. Let AKM be one extremity of a portion, considered free, of a prismatic beam under oblique forces. C is the centre of gravity of the section exposed, and GC the gravity axis ⊥ to the force plane CAK. The stresses acting on the elements of area (each = dF) of the section consist of shears (whose sum = J, the "total shear") in the plane of the section and parallel to the force plane, and of normal stresses parallel to AK and proportional per unit of area to the distances of the dF's on which they act from the neutral axis NC″, real or ideal (ideal in this figure.) Imagine the outermost fibre KA, whose distance from the gravity axis is = e and from the neutral axis = e + a, to be prolonged an amount AA′ whose length by some arbitrary scale, represents the normal stress (tension or compression) to which the dF at A is subjected. Then, if a plane be passed through A′

and the neutral axis NC", the lengths such as mr, parallel to AA', intercepted between this plane and the section itself, represent the stress-intensities (i.e. per unit area) on the respective dF's. (In this particular figure these stresses are all of one kind, all tension or all compression; but if the neutral axis occurs within the limits of the section they will be of opposite kinds on the two sides of NC") Through C', the point determined in A'NC" by the intercept CC' of the centre of gravity, pass a plane A"M"T" parallel to the section itself; it will divide the stress-intensity AA' into two parts p_1 and p_2, and will enable us to express the stress-intensity mr, on any dF at a distance z from the gravity axis GC, in two parts; one part the same for all dF's, the other dependent on z, thus:

$$[\text{Stress-intensity on any } dF] = p_1 + \frac{z}{e} p_2 \quad \ldots\ldots (1)$$

and hence the

$$[\text{actual normal stress on any } dF] = p_1 dF + \frac{z}{e} p_2 dF \ldots (2)$$

For example the stress-intensity on the fibre at T, where $z = -e_1$, will be $p_1 - \frac{e_1}{e} p_2$, and it is now seen how we may find the stress at any dF when p_1 and p_2 have been found. If the distance a, between the neutral and gravity axes is desired, we have, by similar triangles $p_2 : e :: C'C : a$ whence

$$a = \frac{p_1}{p_2} \cdot e \quad \ldots\ldots (3)$$

It is now readily seen, graphically, that the stresses or elastic forces represented by the equal intercepts between the parallel planes AMT and A"M"T", constitute a uniformly distributed normal stress, which will be called the "uniform THRUST", or simply the THRUST (or PULL, as the case may be) of an intensity $= p_1$, and ∴ of an amount $= \int p_1 dF = p_1 \int dF = p_1 F$.

It is also evident that the positive intercepts forming the wedge A"A'G'C' and the negative " " " " " M"M'G'C' form a system of "graded stresses" whose combination (algebraically) with those of the "thrust" shows them to be equivalent to the actual system of normal stresses represented by the small prisms forming the imaginary solid AMT---A'M'T'. It will be shown that these graded stresses constitute a "STRESS COUPLE".

Analytically, the object of this classification of the normal stresses into a thrust and a stress-couple, may be made apparent as follows:

In dealing with the free body KAM Fig. 296, we shall have occasion to sum the components, parallel to the beam, of all forces acting (external and elastic), also those ⊥ to the beam; and also sum their moments about some axis ⊥ to the force plane. Let this axis of moments be GC the gravity-axis of the section, (and not the neutral axis); also take the axis X ∥ to the beam and Y ⊥ to it. Let us see what part the elastic forces will play in these three summations. See Fig. 297, which gives merely a side view. Referring to eq. (2) we see that

$$\left[\text{The } \Sigma X \text{ of the } \begin{matrix}\text{elastic}\\ \text{forces}\end{matrix}\right] = p_1\int_{e_1}^{e} dF + \frac{p_2}{e}\int_{e_1}^{e} z\,dF$$

$$\text{i.e. " " " "} \quad = p_1 F + \frac{p_2}{e} F\bar{z}$$

[see eq. (4) §23] But as the z's are measured from G a gravity axis $\bar{z}$ must be zero. Hence

$$[\text{The } \Sigma X \text{ of Elastic forces}] = p_1 F \equiv \left\{\begin{matrix}\text{the THRUST}\\ \text{or PULL}\end{matrix}\right\} \quad (4)$$

Also,

$$[\text{The } \Sigma Y \text{ of the Elastic forces}] = J \equiv \text{the SHEAR}; \quad (5)$$

while for moments about G [see eq. (1)]

[The Σ(moms.$_G$) of the elastic forces $= \int_{e_1}^{e} (p_1 dF) z + \int_{e_1}^{e} (\frac{z}{e} p_2 dF) z$

$$= p_1 \int_{e_1}^{e} z\, dF + \frac{p_2}{e} \int_{e_1}^{e} z^2 dF$$

and hence finally

[The Σ(moms.$_G$) of the Elastic Forces] $= \dfrac{p_2 I_G}{e} \equiv$ THE MOMENT (6)

where $I_G = \int_{e_1}^{e} z^2 dF$ is the moment of inertia of the section about the gravity axis G, (instead of the neutral axis).

The expression in (6) may be called the MOMENT OF THE STRESS COUPLE, understanding by stress couple a couple to which the graded stresses of Fig. 297 are equivalent. That these graded stresses are equivalent to a couple is shown by the fact that although they are X forces they do not appear in eq. 4, the ΣX, since the sum of the tensions $\left[\frac{p_2}{e}\int_{0}^{e} z\, dF\right]$ equals that of the compressions $\left[\frac{p_2}{e}\int_{e_1}^{0} z\, dF\right]$ in that set of normal stresses.

We have therefore gained these advantages, that of the three quantities J (lbs.), p_1 (lbs. per sq. inch) and p_2 (lbs. per sq. inch) a knowledge of which, with the form of the section, completely determines the stresses in the section, equations (4), (5), and (6) contain only one each, and hence algebraic elimination is unnecessary for finding any one of them; and that the axis of reference of the moment of inertia I is the same axis of the section as was used in former flexure problems.

Another mode of stating eqs. (4) (5) & (6) is this: the sum of the components parallel to the beam of the external forces is balanced by the thrust or pull; those ⊥ to the beam are balanced by the shear, while the sum

of the moments of the external forces about the gravity axis of the section is balanced by that of the stress couple. Notice that the thrust must have no moment about the gravity axis referred to.

The **EQUATION FOR SAFE LOADING**, then, will be this:

$$\left.\begin{array}{l}(a)\cdots(p_1 \pm p_2)\text{max.} \\ \quad\text{or} \\ (b)\cdots(p_1 \pm \frac{e}{e_1}p_2)\text{max}\end{array}\right\}\begin{array}{c}\text{whichever}\\ \text{is}\\ \text{greater}\end{array}\Bigg\} = R' \quad \cdots\cdots(7)$$

(For R' see table in §251. The double sign provides for the cases where p_1 and p_2 are of opposite kinds, one tension the other compression. Of course (p_1+p_2) max. is not the same thing as $[p_1 \text{max.} + p_2 \text{max.}]$. In most cases in practice $e_1 = e$, so that the part (b) of eq. (7) is then unnecessary.

295 a. ELASTIC CURVE WITH OBLIQUE FORCES. (By elastic curve is now meant the locus of the centres of gravity of the sections.) Since the normal stresses in a section differ from those occurring under perpendicular forces only in the addition of a uniform thrust (or pull), whose effect on the short lengths ($=dx$) of fibres between two ~~connective~~ consecutive sections UV and U_0V_0, Fig. 297, is felt equally by all, the location of the centre of curvature R, is not appreciably different from what it would be as determined by the stress couple alone.

Thus (within the elastic limit), strains being proportional to the stresses producing them, if the forces of the stress-couple acted alone, the length $dx = G_0G'$ of a small portion of a fibre at the gravity axis would remain unchanged and the lengthening and shortening of the other fibre-lengths between the two sections U_0V_0 and $U'V'$, originally parallel, would occasion the turning of $U'V'$ through a small angle (relatively to U_0V_0) about G', into the position which it occupies in the fig-

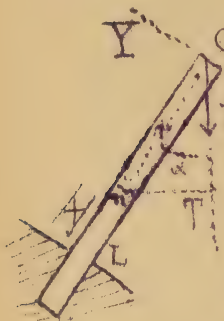


Fig. 298

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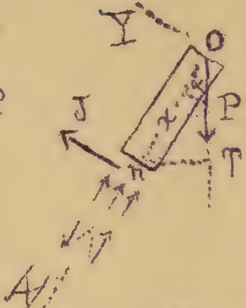

Fig. 299

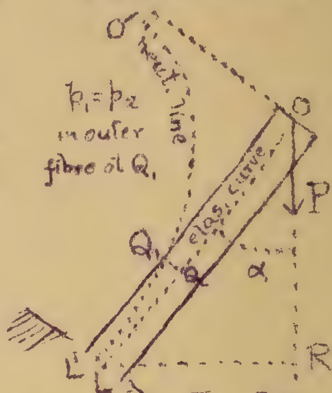

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Fig. 300

§ 296

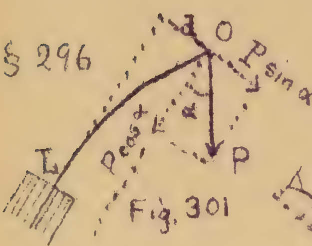


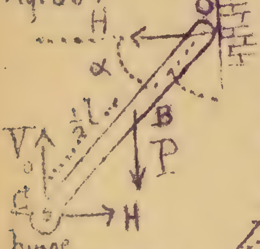
Fig. 301

Fig. 302

§ 297

Fig. 303

Fig. 304



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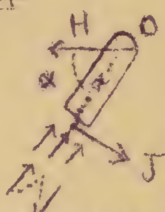

Fig. 305

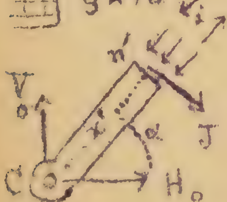

Fig 306

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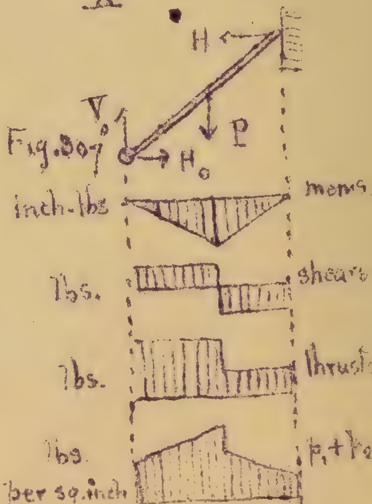


ure (297) and G_0R' would be the radius of curvature. But the effect of the uniform pull (added to that of the couple) is to shift U'V' parallel to itself into the position UV, and hence the radius of curvature of the elastic curve, of which G_0G is an element, is G_0R instead of G_0R'. But the difference between G_0R and G_0R' is very small, being the same, relatively, as the difference between $\overline{G_0G}$ and $\overline{G_0G'}$; for instance, with wrought-iron, even if p, the intensity of the uniform pull were as high as 22000 lbs. per sq. in. [see §203] G_0G would exceed G_0G' by only $\frac{1}{12}$ of one per cent, (=0.0008); hence by using GR' instead of GR as the radius of curvature ρ, an error is introduced of so small an amount as to be negligeable.

But from §231, eqs. (6) and (7), $\frac{EI}{\rho} = EI\frac{d^2y}{dx^2} = M$, the sum of the moments of the external forces; hence for prismatic beams under oblique forces we may still use

$$EI\frac{d^2y}{dx^2} \quad - - - - - - (1)$$

as one form for the Σ (moms.) of the elastic forces of the section about the gravity-axis.

296. OBLIQUE CANTILEVER WITH TERMINAL LOAD. Fig. 298. The "fixing" of the lower end of the beam is its only support. Measure x along the beam from O. Let n be the gravity axis of any section and nT, $= x\sin\alpha$, the length of the perpendicular let fall from n on the line of action of the force P (load). The flexure is so slight that nT is considered to be the same as before the load is allowed to act. [If α were very small, however, it is evident that this assumption would be inadmissible, since then a large proportion of nT would be due to the flexure caused by the load]

Consider nO free, Fig. 299. In accordance with

the preceding paragraph (see eqs (4), (5), and (6)) the elastic forces of the section consist of a shear J, whose value may be obtained by writing $\Sigma Y = 0$ whence

$$J = P \sin\alpha \; ; \qquad (1)$$

of a uniform thrust $= p_1 F$, obtained from $\Sigma X = 0$ viz:

$$P\cos\alpha - p_1 F = 0 \quad \therefore \quad p_1 F = P\cos\alpha \; ; \qquad (2)$$

and of a stress-couple whose moment [which we may write either $\frac{p_2 I}{e}$, or $EI\frac{d^2y}{dx^2}$] is determined from $\Sigma(\text{moms}_n) = 0$ or

$$\frac{p_2 I}{e} - Px\sin\alpha = 0, \quad \text{or} \quad \frac{p_2 I}{e} = Px\sin\alpha \qquad (3)$$

As to the STRENGTH of the beam, we note that the stress-intensity, p_1, of the thrust is the same in all sections, from 0 to L (Fig. 298), and that p_2, the stress-intensity in the outer fibre, (and this is compression if $e = no'$ of fig. 299) due to the stress couple is proportional to x; hence the max. of $[p_1 + p_2]$ will be in the lower outer fibre at L where x is as great as possible, $= l$; and will be a compression, viz.:

$$[p_1 + p_2]\,\text{max.} = P\left[\frac{\cos\alpha}{F} + \frac{l(\sin\alpha)e}{I}\right] \qquad (4)$$

∴ the equation for SAFE LOADING is

$$\frac{R'I}{e} = P\left[\frac{\cos\alpha}{F} + \frac{l(\sin\alpha)e}{I}\right] \qquad (5)$$

since with $e_1 = e$, as will be assumed here, $\left[p_1 - \frac{e_1}{e}p_2\right]_{max}$ can not exceed, numerically, $[p_1 + p_2]$ max. The stress-intensity in the outer fibres along the upper edge of the beam, being $= p_1 - p_2$ (supposing $e_1 = e$) will be compressive at the upper end near O, since there p_2 is small, x being small; but lower down as x grows larger, p_2 increasing, a section may be found (before reaching the point) where $p_2 = p_1$ and where consequently the stress in the outer fibre is zero, or in other words the neutral axis of that

section passes through the outer fibre. In any section above that section the neutral axis is imaginary i.e. is altogether outside the section, while below it, it is within the section, but cannot pass beyond the gravity axis. Thus in Fig. 300, $O'L'$ is the locus of the positions of the neutral axis for successive sections, while OL the axis of the beam is the locus of the gravity axes (or rather of the centres of gravity) of the sections, this latter line forming the "elastic curve" under flexure. As already stated, however, the flexure is to be but slight, and α must not be very small. For instance, if the deflection of O from its position before flexure is of such an amount as to cause the lever-arm LR of P about L to be greater by 10 per cent. then its value ($= l \sin\alpha$) before flexure, the value of p_2 as computed from eq. (3) (with $x = l$) will be less than its true value in the same proportion.

The deflection of O from the tangent at L, by § 237, Fig. 229, is $d = (P \sin\alpha) l^3 \div 3EI$, approximately, putting $P \sin\alpha$ for the P of fig. 229; but this very deflection gives to the other component, $P\cos\alpha$, $\parallel$ to the tangent at L, a lever arm, and consequent moment, about all the gravity axes of all the sections, whence for $\Sigma(\text{moms.}_L) = 0$ we have, (more exactly than from eq. (3) when $x = l$)

$$\frac{p_2 I}{e} = P(\sin\alpha) l + P\cos\alpha \cdot \frac{(P\sin\alpha) l^3}{3EI} \quad \cdots \quad (6)$$

(We have here supposed P replaced by its components $\parallel$ and $\perp$ to the fixed tangent at L see Fig. 301). But even (6) will not give an exact value for p_2 at L; for the lever arm of $P\cos\alpha$, viz. d, is $> (P\sin\alpha) l^3 \div 3EI$, on account of the presence and leverage of $P\cos\alpha$ itself. The true value of d in this case may be obtained by a method similar to that indicated in the next

paragraph.

297. ELASTIC CURVE OF OBLIQUE CANTILEVER WITH TERMINAL LOAD. MORE EXACT SOLUTION. For variety place the cantilever as in Fig. 302 so that the deflection OD = d tends to decrease the moment of P about the gravity axis of any section, n. We may replace P by its X and Y components Fig. 303, ∥ and ⊤ respectively to the fixed tangent line at L. The origin, O, is taken at the free end of the beam. For a free body On, n being any section, we have Σ(moms$_n$) = 0

whence
$$EI\frac{d^2y}{dx^2} = P(\cos\alpha)y - P(\sin\alpha)x \qquad (1)$$

[See eq. (1) §295 a] In this equation the right hand member is evidently (see fig. 303) a negative quantity; this is as it should be, for $EI\,d^2y \div dx^2$ is negative the curve being concave to the axis X in the first quadrant. (It must be noted that the axis X is always to be taken ∥ to the beam, for $EI\,d^2y \div dx^2$ to represent the moment of the stress couple.)

Eq. (1) is not in proper form for taking the x-antiderivative of both members, since one term contains the variable y, an unknown function of x. But by a special solution of Prof. S. W. Robinson (p. 300 of Wood's Resistance of Materials) introducing two constants, which are determined for this problem by the fact that at L for $x = l$, $\frac{dy}{dx} = 0$, and that at O for $x = 0$, $y = 0$, it is proved that the equation of the elastic curve is.

$$\sqrt{\frac{P\cos\alpha}{EI}}\left[e_n^{Bl} + e_n^{-Bl}\right]\left[(\sin\alpha)x - (\cos\alpha)y\right] = \sin\alpha\left[e_n^{Bx} - e_n^{-Bx}\right] \quad (2)$$

in which e_n denotes the NAPERIAN BASE = 2.71828, an abstract number, and B, for brevity, stands for $\sqrt{P\cos\alpha \div EI}$

To find the deflection d, we make $x = l$ in (2).

and solve for y; the result is d.

The uniform thrust at L is $p_1 F = P\cos\alpha$ .... (3)
while the stress intensity p_2 in the outer fibre at L is obtained from the moment equation for the free body LC, viz.:

$$\frac{p_2 I}{e} = P(\sin\alpha)l - P(\cos\alpha)d \qquad (4)$$

in which e = distance of outer fibre from the gravity axis.

The equation for safe loading then is written out by placing the values of p_1, p_2, and d, as derived from equations (2), (3), & (4) in the expression:

$$p_1 + p_2 = R' \qquad (5)$$

To solve the resulting equation for P, in case that is the unknown quantity, can only be accomplished by successive assumptions and approximations, since it occurs transcendentally.

298. INCLINED BEAM WITH HINGE AT ONE END

Fig. 304. Let $e = e_1$. Required the equation for safe loading; also the maximum shear, there being but one load, P, and that in the middle. The vertical wall being smooth its reaction, H, at O is horizontal, while that of the hinge-pin being unknown both in amount and direction is best replaced by its horizontal and vertical components H_0 and V_0, unknown in amount only. Supposing the flexure slight we find these external forces in the same manner as in Prob. 1 § 37 by considering the whole beam free, and obtain

$$H = \frac{P}{2}\cot\alpha\ ; \quad H_0 \text{ also} = \frac{P}{2}\cot\alpha\ ; V_0 = P \quad \ldots \quad (1)$$

For any section n between O and B, we have, from the free body nO, Fig. 305,

$$\text{uniform thrust} = p_1 F = H\cos\alpha \qquad (2)$$

and from $\Sigma(\text{mom.})_n = 0$,

$$\frac{p_2 I}{e} = Hx\sin\alpha \qquad (3)$$

and the shear $= J = H \sin\alpha = \frac{1}{2} P \cos\alpha$ (4)

The max. $(p_1 + p_2)$ to be found on CB [illegible] B, where $x = \frac{1}{2} l$, and is

$$\frac{H \cos\alpha}{F} + \frac{Hle \sin\alpha}{2I} \text{ which} = P\cos\alpha \left[\frac{\cot\alpha}{2F} + \frac{le}{4I} \right] \quad (5)$$

In examining sections on CB let the free body be Cn', fig. 306. Then from Σ(longitud. comps) = 0

$$\text{(the thrust =) } p_1 F = V_0 \sin\alpha + H_0 \cos\alpha \quad (6)'$$

i.e.

$$p_1 F = P \left[\sin\alpha + \frac{1}{2} \cos\alpha \cot\alpha \right] \quad (6)$$

while from $\Sigma(\text{moms.}_{n'}) = 0$;

$$\frac{p_2 I}{e} = V_0 x' \cos\alpha - H_0 x' \sin\alpha \quad (7)'$$

i.e.

$$\frac{p_2 I}{e} = \frac{1}{2} P \cos\alpha \, x' \quad (7)$$

Hence $(p_1 + p_2)$ for sections on CB is greatest when x' is greatest, which is when $x' = \frac{1}{2} l$, x' being limited between $x' = 0$ and $x' = \frac{1}{2} l$, and is

$$(p_1 + p_2)_{max.\ on\ CB} = P\cos\alpha \left[\frac{\tan\alpha + \frac{1}{2}\cot\alpha}{F} + \frac{1}{4} \frac{le}{I} \right] \quad (8)$$

which is evidently greater than the max. $(p_1 + p_2)$ on BC; see eq. (5). Hence the equation for safe loading is

$$R' = P\cos\alpha \left[\frac{\tan\alpha + \frac{1}{2}\cot\alpha}{F} + \frac{1}{4} \frac{le}{I} \right] \quad (9)$$

in which R' is the safe normal stress per square unit of the material.

The shear, J, anywhere on CB, from Σ(transverse comp) = 0 in Fig. 306, is

$$J = V_0 \cos\alpha - H_0 \sin\alpha = \frac{1}{2} P \cos\alpha \quad (10)$$

As showing graphically all the results found, moment, thrust, and shear diagrams are drawn in fig. 307, and also a diagram whose ordinates represent the variation of $(p_1 + p_2)$ along the beam. Each ordinate is placed vertically under the gravity axis of the section to which it refers.

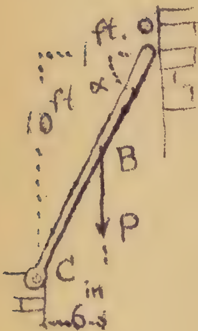


Fig. 308

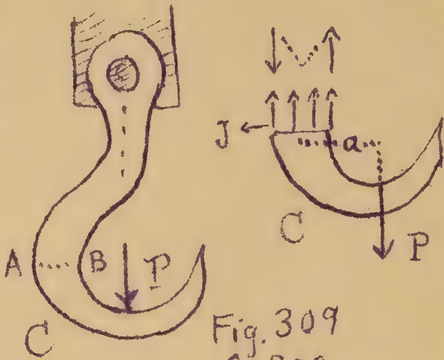
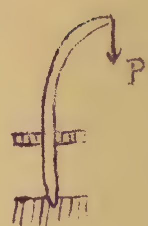
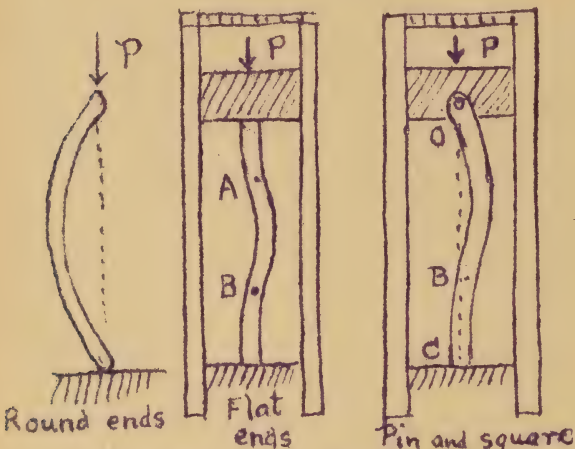


Fig. 309
§ 300

Fig. 310





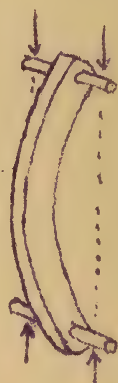


Fig. 312

Fig. 311 § 302

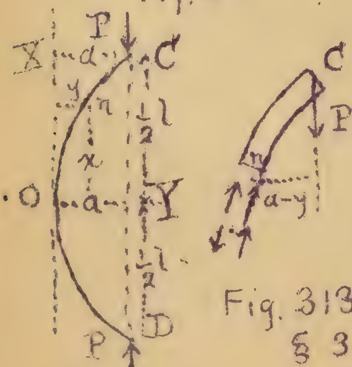


Fig. 313
§ 303

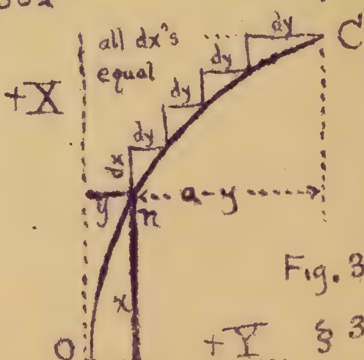


Fig. 314
§ 303

299. NUMERICAL EXAMPLE OF THE FOREGOING. Fig. 308. Let the beam be of wrought iron, the load $P = 1800$ lbs. hanging from the middle. Cross section rectangular $2^{in.}$ by $1^{in.}$, the $2^{in.}$ being parallel to the force-plane. Required the max. normal stress in any outer fibre; also the max. total shear.

The max. stress-intensity will be in the outer fibre in the section just below B and on the upper side, according to § 298, and is given by eq. (8) of that article, in which, see fig. 308, we must substitute (inch pound second system) $P = 1800$ lbs.; $F = 2$ sq. in.; $l = \sqrt{120^2 + 12^2} = 120.6$ in.; $e = 1$ in.; $I = \frac{1}{12}bh^3 = \frac{8}{12} = \frac{2}{3}$ biquad. inches; $\cot\alpha = \frac{1}{10}$; $\cos\alpha = .0996$; and $\tan\alpha = 10$.

$$\therefore \max.(p_1 + p_2) = 1800 \times .0996 \left[\frac{10 + \frac{1}{20}}{2} + \frac{1}{4} \cdot \frac{120.6 \times 1}{2/3}\right] = 9000$$

lbs. per sq. inch, very nearly, compression. This is in the upper outer fibre close under B. In the lower outer fibre just under B we have a tension $= p_2 - p_1 = 7200$ lbs. per sq. inch. (It is here supposed that the beam is secured against yielding sideways.)

300. STRENGTH OF HOOKS. An ordinary hook, see Fig. 309, may be treated as follows: The load being $= P$, if we make a horizontal section at AB, whose gravity axis g is the one, of all sections, furthest removed from the line of action of P, and consider the portion C free, we have the shear $= J =$ zero . . . . . (1)

the uniform pull $= p_1 F = P$ . . . . (2)

while the moment of the stress couple, from $\Sigma(\text{moms}_g) = 0$ is

$$\frac{p_2 I}{e} = Pa \qquad (3)$$

For safe loading $p_1 + p_2$ must $= R'$, i.e.

$$R' = P\left[\frac{1}{F} + \frac{ae}{I}\right] \qquad (4)$$

It is here assumed that . . . , and that the max.

$[p_1 + p_2]$ occurs at AB.

301. CRANE. As an exercise let the student investigate the strength of a crane, such as is shown in fig. 310.

CHAP. VI FLEXURE OF "LONG COLUMNS."

302. DEFINITIONS. By "long column" is meant a straight beam, usually prismatic, which is acted on by two compressive forces, one at each extremity, and whose length is so great compared with its diameter that it gives way (or "fails") by buckling sideways, i.e. by flexure, instead of by crushing or splitting (see §200). The pillars or columns used in buildings, the compression members of bridge trusses and roofs, and the "bents" of a trestle work are the principal practical examples of long columns. That they should be weaker than short blocks of the same material and cross section is quite evident, but their theoretical treatment is much less satisfactory than in other cases of flexure, experiment being very largely relied on not only to determine the physical constants which theory introduces into the formulae referring to them, but even to modify the algebraic form of these formulae, thus rendering them to a certain extent empirical.

303. END CONDITIONS. The strength of a column is largely dependent on whether the ends are free to turn, or are fixed and thus incapable of turning. The former condition is attained by rounding the ends, or providing them with hinges or ball-and-socket-joints; the latter by facing off each end to an accurate plane surface, the bearing on which it rests being plane also, and incapable of turning. In the former condition the column is spoken of as having ROUND ENDS; in the latter as having FIXED ENDS, (or FLAT BASES, or SQUARE ENDS).

Sometimes a column is fixed at one end while the other end is not only round but incapable of lateral deviation from the tangent-line of the first end; this state of end conditions is often spoken of as "PIN AND SQUARE".

Fig. 311 shows examples of these three cases, the flexure being much exaggerated. If the rounding of the ends is produced by a hinge or "pin joint", Fig. 312, both pins lying in the same plane and having immovable bearings at their extremities, the column is to be considered as round-ended as regards flexure in the plane ⊥ to the pins, but as square-ended as regards flexure in the plane containing the axis of the pins.

The "moment of inertia" of the section of a column will be understood to be referred to a gravity axis of the section which is ⊥ to the plane of flexure (and thus corresponds to the "force-plane" spoken of in previous chapters)

303. EULER'S FORMULA. Taking the case of a round-ended column, Fig. 313, assume the middle of the length as an origin, with the axis X tangent to the elastic curve at that point. The flexure being slight, we may use the form $EI\,d^2y \div dx^2$ for the moment of the stress-couple in any section n, remembering that with this notation the axis X must be ∥ to the beam, as in the figure (313). Considering the free body nC, we note that the shear is zero, that the uniform thrust $= P$, and that $\Sigma(\text{moms}_n) = 0$ gives (a being the deflection at O)

$$EI\frac{d^2y}{dx^2} = P(a-y) \quad \text{- - - - - - - (1)}$$

Multiplying each side by dy we have

$$\frac{EI}{dx^2}\,dy\,d^2y = Pa.dy - Py\,dy \quad (2)$$

Since this equation is true for the y, dx, dy and d^2y

of any element of arc of the elastic curve, we may suppose it written out for each element from O where $y=0$, and $dy=0$, up to any element, (where $dy=dy$ and $y=y$) (see Fig. 314) and then write the sum of the left hand members equal to that of the right hand members, remembering that, since dx is assumed constant, $1 \div dx^2$ is a common factor on the left. In other words integrate between O and any point of the curve, n. That is,

$$\frac{EI}{dx^2}\int_{dy=0}^{dy=dy}[dy]\,d[dy] = Pa\int_0^y dy - P\int_0^y y\,dy \quad \cdots \quad (3)$$

The product $dy\,d^2y$ has been written $(dy)\,d\,(dy)$, (for d^2y is the differential or increment of dy) and is of a form like $x\,dx$, or $y\,dy$. Performing the integration we have

$$\frac{EI}{dx^2}\cdot\frac{dy^2}{2} = Pay - P\frac{y^2}{2} \quad (4)$$

which is in a form applicable to any point of the curve, and contains the variables x and y and their increments dx and dy. In order to separate the variables, solve for dx, and we have

$$dx = \sqrt{\frac{EI}{P}}\,\frac{dy}{\sqrt{2ay-y^2}} \quad \text{or} \quad dx = \sqrt{\frac{EI}{P}}\cdot\frac{d\left(\frac{y}{a}\right)}{\sqrt{2\frac{y}{a}-\left(\frac{y}{a}\right)^2}} \cdots (5)$$

$$\therefore \int_0^x dx = \pm\sqrt{\frac{EI}{P}}\int_0^y \frac{d\left(\frac{y}{a}\right)}{\sqrt{2\frac{y}{a}-\left(\frac{y}{a}\right)^2}}\;;\text{ i.e. } x = \pm\sqrt{\frac{EI}{P}}\left(\text{vers.}\sin^{-1}\frac{y}{a}\right) \cdots (6)$$

(6) is the equation of the elastic curve DOC Fig. 313, and contains the deflection a. If P and a are both given y can be computed for a given x and vice versa, and thus the curve traced out, but we would naturally suppose a to depend on P, for in eq. (6) when $x=\frac{1}{2}l$ y should $=a$. Making these substitutions we obtain

$$\tfrac{1}{2}l = \sqrt{\frac{EI}{P}}\left(\text{vers.}\sin^{-1}1.00\right);\text{ i.e. } \tfrac{1}{2}l = \sqrt{\frac{EI}{P}}\cdot\frac{\pi}{2} \quad (7)$$

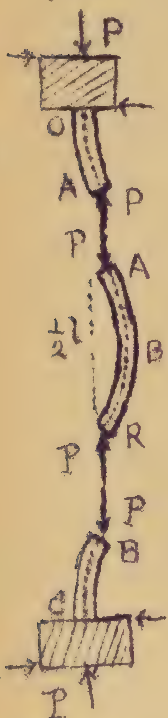


Fig. 315
§ 303

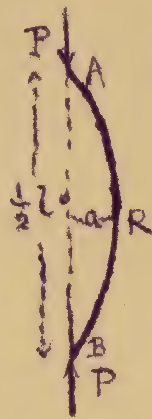


Fig. 316
§ 307

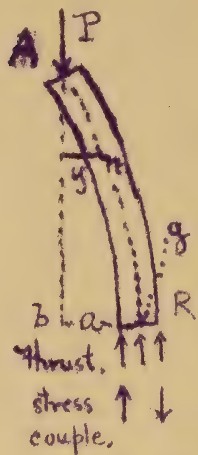
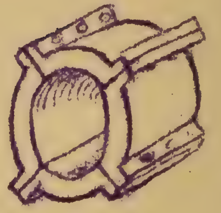
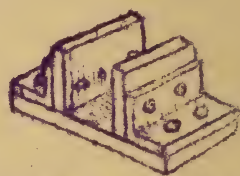


Fig. 317
§ 307



Phoenix Column



Plates and Angles

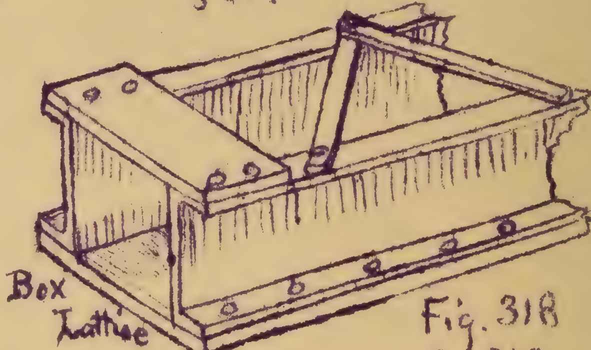


Fig. 318
§ 310

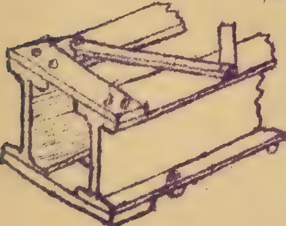


Fig. 319
§ 310

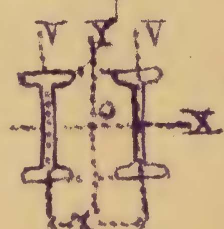


Fig. 320
§ 311

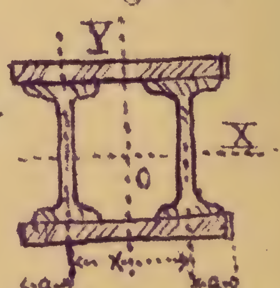


Fig. 321 § 312

Since a has vanished from eq. (7) the value for P obtained from this equation, viz.:

$$P_0 = EI\frac{\pi^2}{l^2} \qquad (8)$$

is independent of a, and

is to be regarded as that force (at each end of the round-ended column in fig. 313) which will hold the column at any small deflection at which it may previously have been set.

In other words if the force is less than P_0 no flexure at all will be produced, and hence P_0 is sometimes called the force producing "incipient flexure". [This is roughly verified by exerting a downward pressure with the hand on the upper end of a flexible rod, (a T-square blade for instance) placed vertically on the floor of a room; the pressure must reach a definite value before a decided buckling takes place, and then a very slight increase of pressure occasions a large increase of deflection]

It is also evident that a force slightly greater than P_0 would very largely increase the deflection, thus gaining for itself so great a lever arm about the middle section as to cause rupture. For this reason eq. (8) may be looked upon as giving the BREAKING LOAD of a column with round ends, and is called Euler's formula.

Referring now to Fig. 311, it will be seen that if the three parts into which the flat-ended column is divided by its two points of inflection A and B are considered free individually in Fig. 315, the forces acting will be as there shown, viz: At the points of inflection there is no stress couple, and no shear, but only a thrust, = P, and hence the portion AB is in the condition of a round-ended column. Also, the tangents to the elastic curves at C and C being preserved vertical by the frictionless guide-blocks and guides (which are introduced here simply as a theoretical method of preventing the ends from turning, but do not interfere with vertical freedom) each is in the same state of flexure as half of AB and under the

same forces. Hence the length AB must = one half the total length l of the flat-ended column. In other words, the breaking load of a round-ended column of length = $\frac{1}{2}l$ is the same as that of a flat-ended column of length = l. Hence for the l of eq. (8) write $\frac{1}{2}l$, and we have as the breaking load of a column with flat-ends

$$P_1 = 4EI\frac{\pi^2}{l^2} \quad \text{------} \quad (9)$$

Similar reasoning, applied to the "pin-and-square" mode of support (in fig 311) where the points of inflection are at E, approximately $\frac{1}{3}l$ from C, and at the extremity O itself, calls for the substitution of $\frac{4}{3}l$ for l in eq. (8), and hence the breaking load of a "pin-and-square" column is

$$P_2 = \frac{9}{4}EI\frac{\pi^2}{l^2} \quad \text{-----------} \quad (10)$$

Comparing eqs. (8), (9), and (10), and calling the value of P_1 (flat-ends) unity, we derive the following statement: The breaking loads of a given column are as the numbers

1	$\frac{9}{16}$	$\frac{1}{4}$	according as its mode of support is
flat-ends	pin-and-square	round-ends	

These ratios are approximately verified in practice.

Euler's Formula [i.e. eq. (8) and those derived from it (9) and (10)] when considered as giving the breaking load is peculiar in this respect, that it contains no reference to the stress per unit of area necessary to rupture the material of the column, but merely assumes that the load producing "incipient flexure", i.e. which produces any bending at all, will eventually break the beam because of the greater and greater lever arm thus gained for itself. In the cantilever of fig. 2?? the bending of the beam does not sensibly affect the lever arm of the load about the wall-section, but with a column, the lever arm of the load about the mid-section is almost entirely due to the deflection produced.

304. EXAMPLE. Euler's formula is only approximately confirmed by experiment. As an example of its use ...

considered as giving the force producing "incipient flexure" it will now be applied to the case of a steel T square blade whose ends are free to turn. Hence we use the rounded-end formula eq. (8) of § 303, with the modulus of elasticity $E = 30\,000\,000$ lbs. per sq. inch. The dimensions are as follows; the length $l = 30$ in., thickness $= \frac{1}{30}$ of an inch, and width $= 2$ inches. The moment of inertia, I, about a gravity axis of the section ∥ to the width (the plane of bending being ∥ to the thickness) is (§ 247)

$I = \frac{1}{12} bh^3 = \frac{1}{12} \times 2 \times \left(\frac{1}{30}\right)^3 = \frac{1}{162000}$ biquad. inches

∴, with $\pi = 22 \div 7$,

$$P_0 = EI\frac{\pi^2}{l^2} = \frac{30\,000\,000}{162\,000} \cdot \frac{22^2}{7^2} \cdot \frac{1}{900} = 2.03 \text{ lbs.}$$

Experiment showed that the force, a very small addition to which caused a large increase of deflection or side-buckling, was about 2 lbs.

305. HODGKINSON'S FORMULAE FOR COLUMNS.

The principal practical use of Euler's formula was to furnish a general form of expression for breaking load, to Eaton Hodgkinson who experimented in England in 1840 upon columns of iron and timber.

According to Euler's formula we have for cylindrical columns, I being $= \frac{1}{4}\pi r^4 = \frac{1}{64}\pi d^4$ (§ 247),

for flat ends .... $P_1 = \frac{1}{64} E\pi^3 . \frac{d^4}{l^2}$

i.e. proportional to the fourth power of the diameter, and inversely as the square of the length. But Hodgkinson's experiments gave for wrought iron cylinders

$P_1 = (\text{const.}) \times \frac{d^{3.55}}{l^2}$; and for cast iron $P_1 = (\text{const.}) \times \frac{d^{3.55}}{l^{1.7}}$

Again, for a square column, whose side $= b$, Euler's formula would give

$$P_1 = \frac{1}{12}\pi^2 E \frac{b^4}{l^2}$$

while Hodgkinson found for square pillars of wood

$$P_1 = (\text{const}) \times \frac{b^4}{l^2}$$

Hence in the case of wood these experiments indicated the same powers for b and l as Euler's formula, but with a different constant factor; while for cast and wrought iron the powers differ slightly from those of Euler.

Hodgkinson's formulae are as follows:

For solid cylindrical <u>cast iron</u> columns, <u>flat-ends</u>

$$\left.\begin{array}{l}\text{Breaking load in TONS}\\ \text{of 2240 lbs. each}\end{array}\right\} = 44.16 \times (d \text{ in inches})^{3.55} \div (l \text{ in feet})^{1.7}$$

For solid cylindrical <u>wrought iron</u> columns, <u>flat-ends</u>.

$$\left.\begin{array}{l}\text{Breaking load in TONS}\\ \text{of 2240 lbs. each}\end{array}\right\} = 134 \times (d \text{ in inches})^{3.55} \div (l \text{ in feet})^{2}$$

For solid square columns of <u>dry oak</u>, <u>flat-ends</u>

$$\left.\begin{array}{l}\text{Breaking load in TONS}\\ \text{of 2240 lbs. each}\end{array}\right\} = 10.95 \times (b \text{ in inches})^{4} \div (l \text{ in feet})^{2}$$

For solid square columns of <u>dry fir</u>, <u>flat-ends</u>

$$\left.\begin{array}{l}\text{Breaking load in TONS}\\ \text{of 2240 lbs. each}\end{array}\right\} = 7.81 \times (b \text{ in inches})^{4} \div (l \text{ in feet})^{2}$$

Hodgkinson found that when the mode of support was "pin and square" the breaking load was about $\frac{1}{2}$ as great; and when the ends were rounded $\frac{1}{3}$ as great as with flat ends. These ratios differ somewhat from the theoretical ones mentioned in §303, just after eq. (10).

Experiment shows that, strictly speaking, pin ends are not equivalent to round ends, but furnish additional strength; for the friction of the pins in their bearings hinders somewhat the turning of the ends. These formulae, (in which d and b denote the diameter and width, and l the length of the column,) not being of homogeneous form, the units to be employed are specified. As the lengths become smaller the value of the breaking load increases very rapidly, until it becomes larger than would be

obtained by using the formula for the crushing resistance of a short block (§201) viz. FC, i.e., the sectional area × the crushing resistance per unit of area.

In such a case the pillar is called a SHORT COLUMN, and the value FC is to be taken as the breaking load. This distinction is necessary in using Hodgkinson's formulae; i.e., the breaking load is the smaller of the two values, FC and that obtained by Hodgkinson's rule.

In present practice Hodgkinson's formulae are little used except for hollow cylindrical iron columns, for which with d_2 and d_1 as the external and internal diameters, we have for flat-ends

$$\left.\begin{array}{l}\text{Breaking load in TONS}\\ \text{of 2240 lbs. each}\end{array}\right\} = \text{Const.} \times \frac{(d_2 \text{ in in.})^{3.55} - (d_1 \text{ in in.})^{3.55}}{(l \text{ in feet})^n}$$

in which the const. = 44.16 for cast iron, and 134 for wrought, while n = 1.7 for cast iron and = 2 for wrought.

306. EXAMPLES OF HODG. FORMULAE.

Example 1. Required the breaking weight of a wrought iron pipe used as a long column, having a length of 12 feet, an internal diameter of 3 in., and an external diam of $3\frac{1}{4}$ inches, the ends having well fitted flat bases.

If we had regard simply to the sectional area of metal, which is F = 1.22 sq. inches, and treated the column as a short block (or short column) we should have for its compressive load at the elastic limit (see table § 203)

$P'' = FC'' = 1.22 \times 24\,000 = 24\,400$ lbs. and the safe load P' might be taken at 16 000 lbs.

But by the last formula of the preceding article we have

$$\left.\begin{array}{l}\text{Breaking load in}\\ \text{tons of 2240 lbs}\end{array}\right\} = 134.0 \times \frac{(3.25)^{3.55} - 3^{3.55}}{12^2} = 14.072$$

i.e. = 14.072 × 2240 = 33720 lbs

DETAIL. [log. 3.25] × 3.55 = 0.511883 × 3.55 = 1.817184

$$[\log. 3.00] \times 3.55 = 0.477121 \times 3.55 = 1.693779$$

The corresponding numbers are 65.6 and 44.4; their difference = 16.2 hence

$$\text{Br. load in long tons} = \frac{134 \times 16.2}{144} = 14.072 \text{ long tons}$$
$$= 33720 \text{ lbs.}$$

With a "factor of safety" (see § 205) of four, we have, as the safe load, P′ = 8430 lbs. This being less than the 16000 lbs. obtained from the "short block" formula, should be adopted.

If the ends were rounded the safe load would be one third of this i.e., would be 2810 lbs. while with pin-and-square end-conditions, we should use one half, or 4215 lbs.

EXAMPLE 2. Required the necessary diameter to be given a solid cylindrical cast iron pillar with flat ends that its safe load may be 13440 lbs. taking 6 as a factor of safety. Let d = the unknown diameter. Using the proper formula in § 305, and hence expressing the breaking load, which is to be six times the given safe load, in long tons we have

$$\frac{13440 \times 6}{2240} = \frac{44.16\,(d \text{ in inches})^{3.55}}{16^{1.7}} \quad (1)$$

$$\text{i.e. } [d \text{ in inches}]^{3.55} = \frac{36 \times 16^{1.7}}{44.16} \quad (2)$$

or

$$\log. d = \frac{1}{3.55}\left[\log. 36 + 1.7 \times \log. 16 - \log. 44.16\right] \quad (3)$$

$$\therefore \log. d = \frac{1}{3.55}[1.958278] = 0.551627 \quad \therefore d = 3.56 \text{ ins.}$$

This result is for flat ends. If the ends were rounded, we would obtain $d = 4.85$ inches.

307. RANKINE'S FORMULA FOR COLUMNS. The formula of this name (sometimes called Gordon's also) has a somewhat more rational basis than Euler's, in that it introduces the maximum normal stress in the

outer fibre and is applicable to a column or block of any length, but still contains assumptions not strictly borne out in theory, thus introducing some co-efficients requiring experimental determination. It may be developed as follows:

Since in the flat-ended column in Fig. 315 the middle portion AB, between the inflection points A and B, is acted on at each end by a thrust $=P$, not accompanied by any shear or stress-couple, it will be simpler to treat that portion alone (Fig. 316) since the thrust and stress-couple induced in the section at R, the middle of AB, will be equal to those at the flat ends, O and C, in Fig. 315. Let a denote the deflection of R from the straight line AB. Now consider the portion AR as a free body in Fig. 317, putting in the elastic forces of the section at R, which may be classified into a uniform thrust $=p_1F$, and a stress couple of moment $=\frac{p_2 I}{e}$, (see § 294). (The shear is evidently zero, from Σ (hor. comps.) $=0$). Here p_1 denotes the uniform pressure (per unit of area) due to the uniform thrust, and p_2 the pressure or tension (per unit of area) in the elastic forces constituting the stress-couple on the outermost element of area, at a distance e from the gravity axis ($\perp$ to plane of flexure) of the section. F is the total area of the section. I is the moment of inertia about the said gravity axis, g.

$$\Sigma(\text{vert. comps.}) = 0 \quad \text{gives } P = p_1 F \quad \text{- - - - (1)}$$
$$\Sigma(\text{moms.}_g) = 0 \quad \text{gives } Pa = \frac{p_2 I}{e} \quad \text{- - - (2)}$$

For any section, n, between A and R, we would evidently have the same p_1 as at R, but a smaller p_2, since $Py < Pa$ while e, I, and F, do not change, the column being prismatic. Hence the max. (p_1+p_2) is on the concave edge at R and for safety should be

no more than $C \div n$, where C is the Modulus of Crushing (§201) and n is a "factor of safety". Solving (1) and (2) for p_1 and p_2, and putting their sum $= C \div n$, we have

$$\frac{P}{F} + \frac{Pae}{I} = \frac{C}{n} \quad \text{------- (3)}$$

We might now solve for P and call it the safe load, but it is customary to present the formula in a form for giving the <u>breaking load</u>, the factor of safety being applied afterward. Hence we shall make $n = 1$, and solve for P, calling it then the breaking load. Now the deflection a is unknown, but supposing it to be proportional approximately to $l^2 \div e$, we may write $ae = \beta l^2$, β being an abstract number dependent on experiment. We may also write, for convenience, $I = Fk^2$, k being the radius of gyration (see § 85). Hence, finally, we have from eq. (3)

$$\left.\begin{array}{l}\text{Breaking load}\\ \text{for flat ends}\end{array}\right\} = P_1 = \frac{FC}{1 + \beta\frac{l^2}{k^2}} \quad \text{---------- (4)}$$

This is known as Rankine's formula.

By the same reasoning as in § 303, for a round-ended column we substitute $2l$ for l; for a pin-and-square column $\frac{4}{3}l$ for l; and $\therefore$ obtain

$$\left.\begin{array}{l}\text{Breaking load}\\ \text{for a round-ended column}\end{array}\right\} = P_0 = \frac{FC}{1 + 4\beta\frac{l^2}{k^2}} \quad \text{------ (5)}$$

$$\left.\begin{array}{l}\text{Breaking load for}\\ \text{a pin-and-square column}\end{array}\right\} = P_2 = \frac{FC}{1 + \frac{16}{9}\beta\frac{l^2}{k^2}} \quad \text{---- (6)}$$

These formulae, (4), (5), and (6), unlike Hodgkinson's are of <u>homogeneous form</u>. Any convenient system of units may therefore be used in them.

Rankine gives the following values for C and β, to be used in these formulae. These are based on Hodgkinson's experiments.

	Cast Iron	Wro't Iron	Timber
C in lbs per sq. in.	80 000	36 000	7200
$\mathfrak{z}$ (abstract number)	$\frac{1}{6400}$	$\frac{1}{36000}$	$\frac{1}{3000}$

If these numerical values of C are used F must be expressed in SQ. INCHES and P in POUNDS. Rankine recommends 4 as a factor of safety for iron in quiescent structures, 5 under moving loads; 10 for timber.

308. EXAMPLES, USING RANKINE'S FORMULA.

Example 1. Take the same data for a wrought iron pipe used as a column as in example 1. § 306; i.e. $l = 12$ ft. $= 144$ inches, $F = \frac{1}{4}\left[\pi\left(3\frac{1}{4}\right)^2 - \pi\, 3^2\right] = 1.227$ sq. inches, while k^2 for a narrow circular ring like the present section may be put $= \frac{1}{2}\left(1\frac{5}{8}\right)^2$ (see § 98) sq. inches. With these values, and $C = 36\,000$ lbs. per sq. in., and $\mathfrak{z} = \frac{1}{36000}$ (for wrought-iron), we have from eq. (4), for flat ends,

$$P_1 = \frac{1.227 \times 36000}{1 + \frac{1}{36000} \cdot \frac{(144)^2}{\frac{1}{2}[1.625]^2}} = 30743.6 \text{ lbs.} \quad \text{---} \quad (1)$$

This being the breaking load, the safe load may be taken $= \frac{1}{4}$ or $\frac{1}{5}$ of 30743.6 lbs., according as the structure of which the column is a member is quiescent or subject to vibration from moving loads. By Hodgkinson's formula 33 720 lbs. was obtained as a breaking load in this case (§ 306).

For rounded ends we should obtain (eq. 5)

$$P_0 = 16100. \text{ lbs. as break. load} \qquad (2)$$

and for pin-and-square, eq. (6)

$$P_2 = 24900 \text{ lbs. as break load} \qquad (3)$$

EXAMPLE 2. (Same as Example 2, § 306) Required

by Rankine's formula the necessary diameter, d, to be given a solid cylindrical cast-iron pillar, 16 ft. in length, with rounded ends, that its safe load may be six long tons (i.e. of 2240 lbs. each) taking 6 as a factor of safety. $F = \frac{\pi d^2}{4}$, while the value of k^2 is thus obtained: From § 247, I for a full circle about its diameter $= \frac{1}{4}\pi r^4 = \pi r^2 \cdot \frac{1}{4} r^2 \therefore k^2 = \frac{1}{4} r^2 = \frac{1}{16} d^2$. Hence eq. (5) of § 307 becomes

$$P_0 = \frac{\frac{1}{4}\pi d^2 C}{1 + 4\beta \frac{16 l^2}{d^2}} \quad - - - - \quad (1)$$

P_0 the breaking load is to be $= 6 \times 6 \times 2240$ lbs., C for cast iron is 80 000 lbs. per sq. inch, while β (abstract number) $= \frac{1}{6400}$. Solving for d we have first the biquadratic equation:

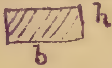
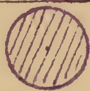
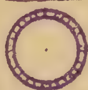
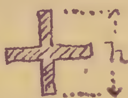
$$d^4 - \frac{28 \times 6 \times 6 \times 2240}{22 \times 80\,000} d^2 = \frac{28 \times 6 \times 6 \times 2240 \times 16^2 \times 12^2 \times 4}{22 \times 80000 \times 400}$$

whence $d^2 = 0.641(1 \pm 33.92)$, and taking the upper sign, finally, $d = \sqrt{22.4} = 4.73$ inches

(By Hodgkinson's rule we obtained 4.85 inches)

309. RADII OF GYRATION. The following table, taken from p. 523 of Rankine's Civil Engineering, gives values of k^2, the square of the <u>least</u> radius of gyration of the given cross section about a gravity-axis. By giving the <u>least</u> value of k^2 it is implied that the plane of flexure is not determined by the end-conditions of the column; i.e. that the column has either flat ends or round ends. If either end (or both) is a <u>pin-joint</u> the column may need to be treated as having a flat-end as regards flexure in an axial plane passing through the pin, if the bearings of the pin are firm, while as regards flexure in a plane perpendicular to the pin it is round-ended at that extremity.

In the case of a "Thin cell" the value of k^2 is strictly true for metal infinitely thin; still, if that thickness does not exceed 1/8 of the exterior diameter, the form given is sufficiently near for practical purposes; similar statements apply to the branching forms.

Solid Rectangle h = least side		$k^2 = \frac{1}{12} h^2$
Thin Square Cell Side = h		$k^2 = \frac{1}{6} h^2$
Thin Rectangular Cell h = least side		$k^2 = \frac{h^2}{12} \cdot \frac{h + 3b}{h + b}$
Solid Circular Section diameter = d		$k^2 = \frac{1}{16} d^2$
Thin Circular Cell Exterior diam. = d		$k^2 = \frac{1}{8} d^2$
Angle-Iron of Equal ribs		$k^2 = \frac{1}{24} \cdot b^2$
Angle-Iron of unequal ribs		$k^2 = \frac{1}{12} \cdot \frac{b^2 h^2}{12(b^2 + h^2)}$
Cross of equal arms		$k^2 = \frac{1}{24} h^2$
I-Beam as a pillar. Let area of web = B " " " both flanges = A		$k^2 = \frac{b^2}{12} \cdot \frac{A}{A+B}$
Channel Iron. Let area of web = B; of flanges = A (both). h is from edge of flange to middle of web.		$k^2 = h^2 \left[\frac{A}{12(A+B)} + \frac{AB}{4(A+B)^2} \right]$

310. BUILT COLUMNS. The "compression members" of wrought-iron bridge trusses are generally composed of several pieces riveted together, the most common forms being the Phoenix column (ring-shaped, in segments,) and combinations of channels, plates, and lattice, some of which are shown in Fig. 318.

Experiments on full size columns of these kinds were made by the U.S. Testing Board at the Watertown Arsenal about 1880.

The Phoenix columns ranged from 8 in. to 28 ft. in length, and from 1 to 42 in the value of the ratio of length to diameter. The breaking loads were found to be somewhat in excess of the values computed from Rankine's formula; from 10 to 40 per cent. excess. In the pocket-book issued by the Phoenix company they give the following formula (wrought iron) for their columns.

$$\left.\begin{array}{l}\text{Breaking load in lbs.}\\ \text{for flat-ended columns}\end{array}\right\} = \frac{50\,000\,F}{1 + \dfrac{l^2}{3000\,h^2}} \quad \cdots \quad (1)$$

where F = area in sq.in., l = length, and h = diameter.

Many different formulae have been proposed by different engineers to satisfy these and other recent experiments on columns, but all are of the general form of Rankine's. For instance Mr. Bouscaren of the Keystone Bridge Co. claims that the strength of Phoenix columns is best given by the formula

$$\left.\begin{array}{l}\text{Breaking load in lbs.}\\ \text{for flat ends}\end{array}\right\} = \frac{38000\,F}{1 + \dfrac{l^2}{100000\,k^2}} \quad \cdots \quad (2)$$

(F must be in sq. inches)

The moments of inertia, I, and thence the value of $k^2 = I \div F$, for such sections as those given in Fig. 318 may be found by the rules of §§ 85–93, (see also § 258).

311. MOMENT OF INERTIA OF BUILT COLUMN. EXAMPLE. It is proposed to form a column by joining two I-beams by lattice work, Fig. 319. (While the lattice work is relied on to cause the beams to act together as one piece, it is not regarded in estimating the area F, or the moment of inertia I, of the cross section). It is also required to find the proper distance apart $= x$, Fig. 320, at which these beams must be placed, from centre to centre of webs, that the liability to flexure shall be equal in all axial planes, i.e. that the I of the compound section shall be the same about all gravity axes. Fig. 320. This condition will be fulfilled if I_Y can be made $= I_X$, O being the centre of gravity of the compound section, and X perpendicular to the ∥ webs of the two equal I-beams.

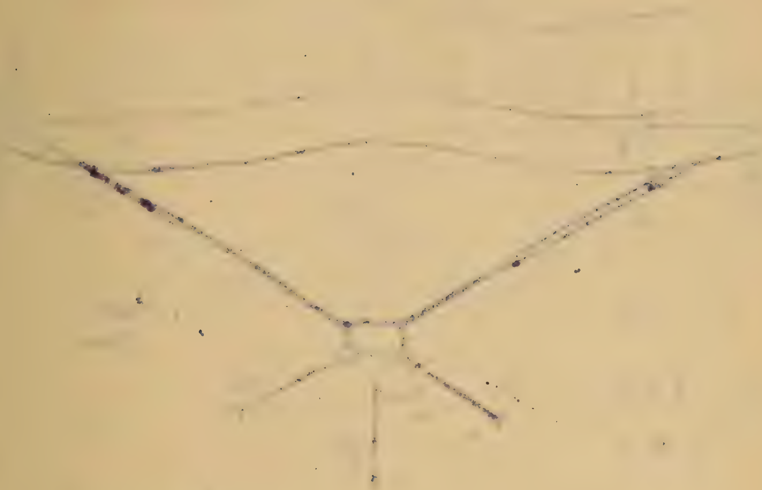
Let $F' =$ the sectional area of one of the I-beams, I'_Y (see Fig. 320) its moment of inertia about its web-axis, I' that about an axis ⊥ to web. (These quantities can be found in the hand-book of the Iron company, for each size of rolled beam).

Then the total $I_X = 2I'_X$; and total $I_Y = 2\left[I'_Y + F'\left(\frac{x}{2}\right)^2\right]$ (see § 88 eq. 4) If these are to be equal, we write them so and solve for x, obtaining

$$x = \sqrt{\frac{4[I'_X - I'_Y]}{F'}} \quad \text{- - - - - - - (1)}$$

312. NUMERICALLY, suppose each girder to be a $10\frac{1}{2}$ inch light I-beam, 105 lbs. per yard, of the N. J. Steel and Iron Co., in whose hand book we find that for this beam $I'_X = 185.6$ biquad. inches and $I_Y = 9.43$ biquad. inches, while $F' = 10.44$ sq. inches. With these values in eq. (1) we have

$$x = \sqrt{\frac{4(185.6 - 9.43)}{10.44}} = \sqrt{67.5} = 8.21 \text{ inches}$$



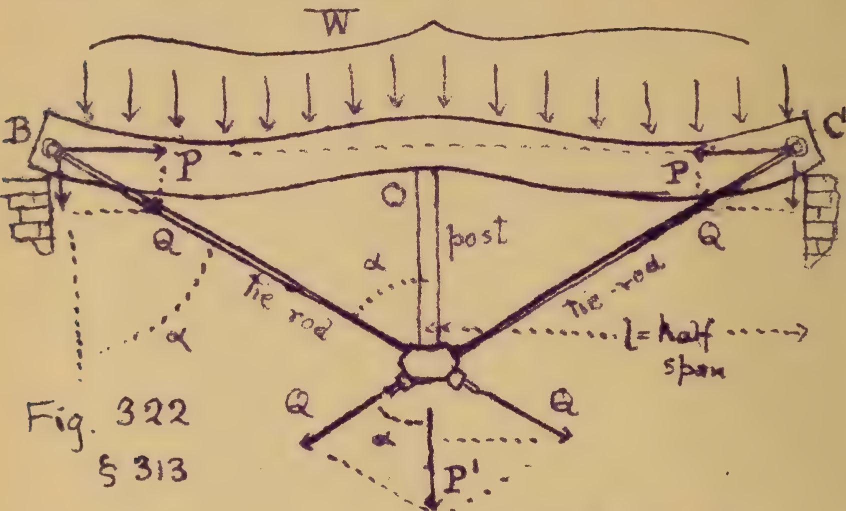


Fig. 322
§ 313

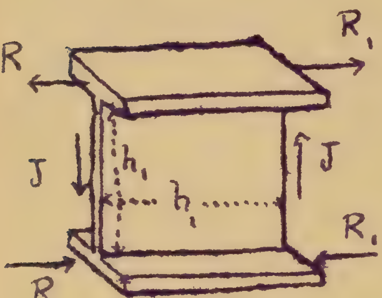


Fig. 323 § 314

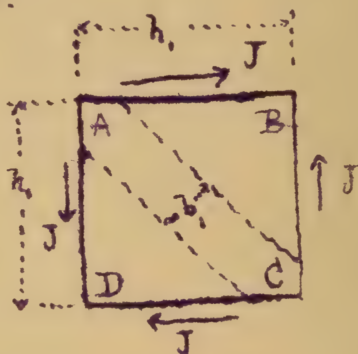


Fig. 324

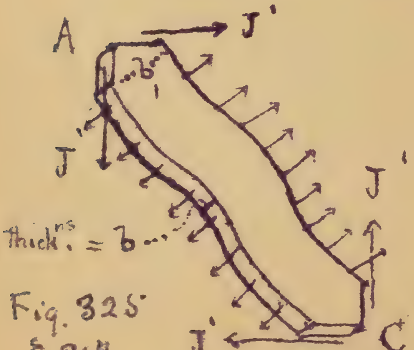


Fig. 325
§ 314

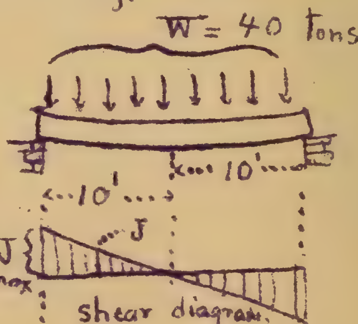


Fig. 326 § 314

The square of the radius of gyration will be

$$k^2 = 2I_X \div 2F' = 371.2 \div 20.88 = 17.7 \text{ sq.in.} \quad \text{---}(2)$$

and is the same for any gravity axis (see § 89).

As an ADDITIONAL EXAMPLE, suppose the two I-beams united by plates instead of lattice. Let the thickness of the plate be $= t$ Fig. 321. Neglect the rivet-holes. The distance a is known from the hand-book. The student may derive a formula for x, imposing the condition that (total I_X) $= I_Y$.

313. TRUSSED GIRDERS.

When a horizontal beam is trussed in the manner indicated in Fig. 322, with a single post or strut under the middle and two tie-rods, it is subjected to a longitudinal compression due to the tension of the tie-rods, and hence to a certain extent resists as a column, the plane of whose flexure is vertical, (since we shall here suppose the beam supported laterally.) Taking the case of uniform loading, (total load $= W$) and supposing the tie rods screwed up (by sleeve nuts) until the top of the post is on a level with the piers, we know that the pressure between the post and the beam is $P' = \frac{5}{8}W$ (see § 273) Hence by the ▱ of forces (see fig. 322) the tension in each tie-rod is

$$Q = \frac{P'}{2\cos\alpha} = \frac{5}{16} \cdot \frac{W}{\cos\alpha}$$

At each pier the horizontal component of Q is

$$P = Q\sin\alpha = \frac{5}{16} W \tan\alpha \quad \text{------} \quad (1)$$

Hence we are to consider the half-beam BO as a "pin-and-square" column under a compressive force $P = \frac{5}{16} W \tan\alpha$, as well as a portion of a continuous girder over three equidistant supports at the same level and bearing a uniform load W. In the outer fibre of the dangerous section, O, (see also § 273 and Fig. 278) the compression per sq. inch due to both these straining ac-

tions must not exceed a safe limit, R', (see § 251). In eq. (6) § 307, where P_2 is the breaking force for a pin-and-square column, the greatest stress in any outer fibre $= C$ (= the Modulus of Crushing) per unit of area. If then we write $p_{col.}$ instead of C in that equation, and $\frac{5}{16} W \tan\alpha$ instead of P_2 we have

$$\left\{\begin{matrix}\text{max. stress due}\\ \text{to column action}\end{matrix}\right\} = p_{col.} = \frac{5}{16}\cdot\frac{W\tan\alpha}{F}\left[1+\frac{16}{9}\cdot\beta\cdot\frac{l^2}{k^2}\right];$$

while from § 273 eq. (7) we have

$$\left\{\begin{matrix}\text{max. stress due}\\ \text{to girder action}\end{matrix}\right\} = p_g = \frac{1}{8}\,\frac{Wle}{I} = \frac{1}{8}\cdot\frac{Wle}{Fk^2}$$

By writing $p_{col.} + p_g = R'$ = a safe value of compression per unit-area, we have <u>the equation for safe loading</u>

$$W\left[5\tan\alpha\left(1+\frac{16}{9}\cdot\beta\cdot\frac{l^2}{k^2}\right)+\frac{2le}{k^2}\right] = 16FR' \quad \text{- - - (1)}$$

Here l = the <u>half-span</u> OB, Fig. 322, e = the distance of outer fibre from the <u>horizontal</u> gravity axis of the cross section, k^2 = the radius of gyration of the section referred to the same axis, while F = area of section. β should be taken from the end of § 307.

EXAMPLE. If the span is 30 ft. = 360 in., the girder a 15 inch heavy I-beam of wrought iron, 200 lbs. to the yard, in which $e = \frac{1}{2}$ of $15 = 7\frac{1}{2}$ inches, $F = 20$ sq. in., and $k^2 = 35.3$ sq. inches (taken from the Trenton Co's hand-book), required the safe load W, the strut being 5 ft. long. From § 307, $\beta = 1:36000$; $\tan\alpha = 15 \div 5 = 3.00$. Hence, using the units pound and inch throughout, and putting $R' = 12000$ lbs. per sq. in. = max. allowable compression stress, we have from eq. (1)

$$W = \frac{16\times 20\times 12000}{15\left[1+\frac{16}{9}\cdot\frac{1}{36000}\cdot\frac{(180)^2}{35.3}\right]+\frac{2\times 180\times 7\frac{1}{2}}{35.3}} = 41650 \text{ lbs}$$

i.e. 39650 lbs. besides the weight of the beam.

[Let the student design the tie-rods (and the strut)].

314. BUCKLING OF WEB-PLATES IN BUILT GIRDERS. In § 257 mention was made of the fact that very high web plates in built beams, such as I beams and box-girders, might need to be stiffened by riveting T irons on the sides of the web. (The girders here spoken of are horizontal ones such as might be used for carrying a railroad over a short span of 20 to 30 feet)

An approximate method of determining whether such stiffening is needed to prevent lateral buckling of the web may be based upon Rankine's formula for a long column and will now be given.

In Fig. 323 we have, free, a portion of a bent I-beam, between two vertical sections at a distance apart $= h =$ the height of the web. In such a beam under forces ⌉ to its axis it has been proved (§ 256) that we may consider the web to sustain all the shear, J, at any section, and the flanges to take all the tension and compression, which form the "stress-couple" of the section. These couples and the two shears are shown in Fig. 323, for the two exposed sections. There is supposed to be no load on this portion of the beam, hence the shears at the two ends are equal. Now the shear acting between each flange and the horizontal edge of the web is equal in intensity per square inch to that in the vertical edge of the web; hence if the web alone, of Fig. 323, is shown as a free body in Fig. 324, we must insert two horizontal forces $= J$, in opposite directions, on its upper and lower edges. Each of these $= J$ since we have taken a horizontal length $h_1 =$ height of web. In this figure, 324, we notice that the effect of the acting forces is to lengthen the diagonal BD and shorten the diagonal AC, both of these diagonals making an angle of $45°$ with the horizontal.

Let us now consider this buckling tendency along AC, by treating as free the strip AC of small width $= b_1$. This is shown in Fig. 325. The only forces acting in the direction of its length AC are the horizontal components of the four forces J' at the extremities. We may therefore treat the strip as a long column of a length $l = h_1\sqrt{2}$, of a sectional area $F = bb_1$, (where b is the thickness of the web plate), having a value of $k^2 = \frac{1}{12}b^2$ (see § 309), and with fixed (or flat) ends. Now the sum of the longitudinal components of the two J''s at A is $Q = 2J'\frac{1}{2}\sqrt{2} = J'\sqrt{2}$; but J' itself $= \frac{J}{bh_1} \cdot b\frac{1}{2}b_1\sqrt{2}$, since the small rectangle on which J' acts has an area $= b\frac{1}{2}b_1\sqrt{2}$, and the shearing stress on it has an intensity of $(J \div bh_1)$ per unit of area. Hence the longitudinal force at each end of this long column is

$$Q = \frac{b_1}{h_1}J \quad \text{- - - - - - - -} \quad (1)$$

According to eq. (4) and the table in § 307, the safe load (factor of safety = 4) for a wrought-iron column of this form, with flat ends, would be

$$P_1 = \frac{\frac{1}{4}bb_1\,36000}{1 + \frac{1}{36000}\cdot\frac{2h_1^2}{\frac{1}{12}b^2}} = \frac{9000\,bb_1}{1 + \frac{1}{1500}\cdot\frac{h_1^2}{b^2}} \quad \text{- - - -} \quad (2)$$

If, then, in any particular locality of the girder (of wrought-iron) we find that Q is $> P_1$, i.e.

$$\text{if } \frac{J}{h_1} \text{ is } > \frac{9000\,b}{1 + \frac{1}{1500}\cdot\frac{h_1^2}{b^2}} \quad \text{- - - - (see N.B.) - - - -} \quad (3)$$

then vertical stiffeners will be required laterally.
[N.B. Eq. (3) is not homogeneous but requires the use of the POUND and INCH].

When these are required, they are generally placed at intervals equal to h_1, (the depth of web), along that part of the girder where Q is $> P_1$.

EXAMPLE. Fig. 326. Will stiffening pieces be required

in a built girder of 20 feet span, bearing a uniform load of 40 tons, and having a web 24 in. deep and 3/8 in. thick?

From § 242 we know that the greatest shear, $J_{max.}$, is close to either pier, and hence we investigate that part of the girder first.

$J_{max.} = \frac{1}{2} W = 20$ tons $= 40000$ lbs.

∴ (inch & lb.), from (2),

$$\frac{J}{h_1} = \frac{40000}{24} = 1666.6 \quad \text{-------} \quad (4)$$

while, from (3), (inch and pound),

$$\frac{9000 \times \frac{3}{8}}{1 + \frac{1}{1500} \cdot \frac{24^2}{(3/8)^2}} = 905.0 \quad \text{-------} \quad (5)$$

Hence stiffening pieces will be needed near the extremities of the girder. Also, since the shear for this case of loading diminishes slowly toward zero at the middle they will be needed from each end up to a distance of $\frac{1666}{905}$ of 10 ft. from the middle.

CHAP. VII.

Linear Arches (of Blockwork).

315. A BLOCKWORK ARCH, is a structure, spanning an opening or gap, depending, for stability, upon the resistance to compression of its blocks, or voussoirs, the material of which, such as stone or brick, is not suitable for sustaining a tensile strain. Above the voussoirs is usually placed a load of some character, (e.g. a roadway,) whose pressure upon the voussoirs will be considered as vertical, only. This condition is not fully

realized in practice, unless the load is of cut stone, with vertical and horizontal joints resting upon voussoirs of corresponding shape (see Fig. 327) but sufficiently so to warrant its assumption in theory. *Symmetry of form* about a vertical axis will also be assumed in the following treatment.

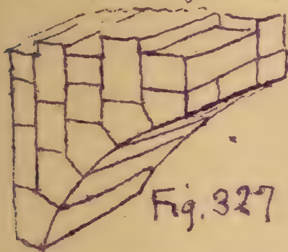


Fig. 327

316. LINEAR ARCHES. For purposes of theoretical discussion the voussoirs of Fig. 327 may be considered to become infinitely small and infinite in number, thus forming a "linear arch", while retaining the same shapes, their depth ⊥ to the paper being assumed = unity that it may not appear in the formulae. The joints between them are ⊥ to the curve of the arch, i.e., adjacent voussoirs can exert pressure on each other only in the direction of the tangent-line to that curve

317. INVERTED CATENARY, OR LINEAR ARCH SUSTAINING ITS OWN WEIGHT ALONE. Suppose the infinitely small voussoirs to have weight, uniformly distributed along the CURVE, weighing q lbs. per running foot. The equilibrium of such a structure, Fig. 328, is of course unstable but theoretically possible. Required the form of the curve when equilibrium exists. The conditions of equilibrium are, obviously: 1st The thrust or mutual pressure T between any two adjacent voussoirs at any point, A, of the curve must be tangent to the curve; and 2ndly, Considering a portion BA as a free body,

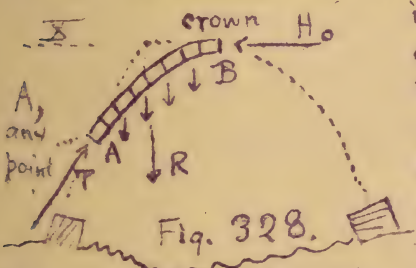


Fig. 328.

the resultant of H_o at B and T at A must balance R the resultant of the ∥ vertical forces (i.e. weights of the elementary voussoirs) acting between B and A.

But the conditions of equilibrium of a flexible, inextensible and uniformly cord or chain are the very same (weights uniform along the curve) the forces being reversed in direction Fig. 329. Instead of compression we have tension, while the ∥ vertical forces act toward, instead of away from, the axis X. Hence the curve of equilibrium of Fig. 328 is an inverted catenary (see § 48) whose equation is

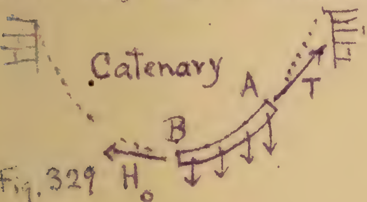


Fig. 329

$$y + c = \frac{1}{2}c\left[e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right] \quad \ldots (1.)$$

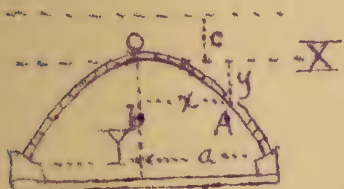


Fig. 330.

See Fig. 330. $e = 2.71828$ the Naperian Base. The parameter c may be determined by putting $x = a$ the half span, and $y = b$ the rise, then solving for c by successive approximations. The horizontal thrust, or H_o, is $= qc$, while if s = length of arch OA, along the curve, the thrust T at any point A is

$$T = \sqrt{H_o^2 + q^2 s^2} \quad \ldots\ldots (2.)$$

From the foregoing it may be inferred that a series of voussoirs of finite dimensions, arranged so as to contain the catenary curve, with joints ⊥ to that curve and of equal weights for equal lengths of arc will be in equilibrium, and moreover in stable equilibrium on account of friction, and and the finite width of the joints. see Fig. 331


Fig 331

318. LINEAR ARCHES under GIVEN LOADING. The linear arches to be considered further will be treated as without weight themselves but as bearing vertically pressing loads (each rowsover its own)

PROBLEM. Given the form of the linear arch itself, it is required to find the law of vertical depth of loading under which the given linear arch will be in equilibrium. Fig. 332, given the curve ABC i.e. the linear arch itself, required the form of the curve MON, or upper limit of loading, such that the linear arch ABC shall be in equilibrium under the loads lying between the two curves.

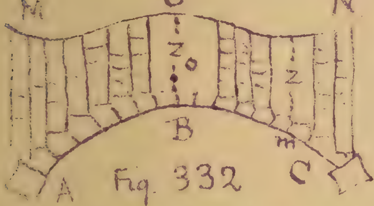
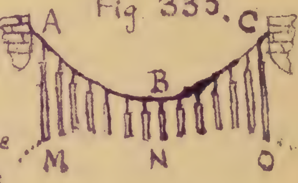


Fig. 332

The loading is supposed homogeneous and of unit depth ⊥ to paper; so that the ordinates z between the two curves are proportional to the load per horizontal linear unit. Assume a height of load z_0 at the crown, at pleasure; then required the z of any point m as a function of z_0 and the curve ABC.

PRACTICAL SOLUTION. Since a linear arch under vertical pressures is nothing more than the inversion of the curve assumed by a cord loaded in the same way, this problem might be solved mechanically by experimenting with a light cord, Fig. 333, to which are hung other heavy cords, or bars of uniform weight per unit length and at equal horizontal distances apart when in equilibrium. By varying the lengths of the bars, and their points of attachment, we may finally find the curve sought, MNO.

Fig. 333.



ANALYTICAL SOLUTION. Consider the structure in Fig. 334. A number of rods of finite length are in equilibrium, and bear the weights P, P', etc at the con-

meeting joints, each piece exerting a thrust T against the adjacent joint. The joint A (imagined separated from the contiguous rods and hence free) is held in equilibrium by the vertical force P (a load) and the two thrusts T and T', making angles = θ and θ' with the vertical, Fig. 335 shows the joint A free. From Σ(horizontal compsi) = 0 we have

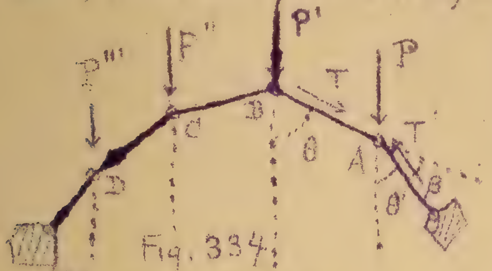


Fig. 334.

Fig. 335

$$T \sin\theta = T' \sin\theta'$$

That is, $T \sin\theta$ is the same for all the rods and hence $= H_0$ the thrust at the crown, where the rod is horizontal (if any there). Hence

$$T = \frac{H_0}{\sin\theta} \quad \ldots\ldots (1.)$$

Now draw a line ms ⊥ to T and write Σ(compons. to ms) = 0, whence $P \sin\theta' = T \sin\beta$, and [see (1)]

$$\therefore \quad P = \frac{H_0 \sin\beta}{\sin\theta \sin\theta'} \quad \ldots\ldots\ldots (2.)$$

Let the rods of Fig. 334 become infinitely small and infinite in number and the load continuous. Each rod becomes = ds an element of the linear arch. β is the angle between two consecutive ds's, θ is the angle between the tangent line and the vertical, while P becomes the load resting on a single dx, or horizontal distance between the middles of the two ds's. That is, Fig. 336, if γ = weight of a cubic unit of the loading, $P = \gamma z dx$ (The lamina of arch and load considered is unity ⊥ to paper, in thickness.). H_0 = a constant = thrust at crown O; θ = θ', and

$\sin\beta = ds \div \rho$, (since the angle between two consecutive tangents is = that between two " " radii of curvature) Hence eq. (2) becomes

$$\gamma z\, dx = \frac{H_0\, ds}{\rho\, \sin^2\theta}\,; \quad \text{but } dx = ds\,\sin\theta,$$

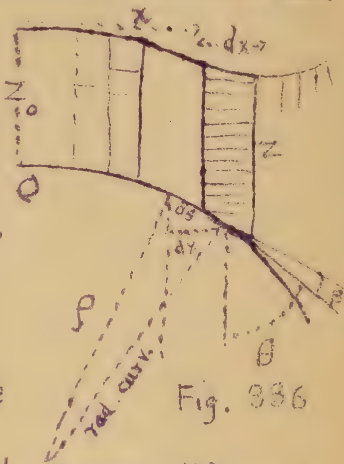


Fig. 336

$$\therefore\ \gamma z = \frac{H_0}{\rho\,\sin^3\theta} \quad \ldots\ldots (3.)$$

Call the radius of curvature at the crown ρ_0, and since there $z = z_0$ and $\theta = 90°$ (3) gives $\gamma z_0 \rho_0 = H_0$; hence (3) may be written

$$z = \frac{z_0\,\rho_0}{\rho\,\sin^3\theta} \quad \ldots\ldots \quad (4)$$

This the law of vertical depth of loading required. For a point of the linear arch where the tangent line is vertical, $\sin\theta = 0$ and z would $= \infty$; i.e. the load would be infinitely high. Hence in practice a full semi-circle for instance could not be used as a linear arch.

319. CIRCULAR ARC AS LINEAR ARCH.

As an example of the preceding problem let us apply eq. (4) to a circular arc, Fig. 337, as a linear arch. Since for a circle ρ is constant and $= r$, eq. (4) reduces to

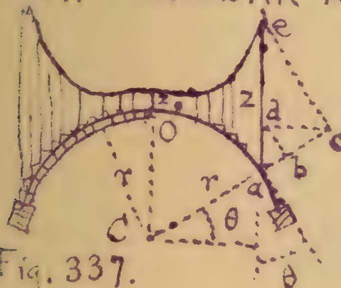


Fig. 337.

$$z = \frac{z_0}{\sin^3\theta} \quad \ldots\ldots (5.)$$

Hence the depths of loading must vary inversely as the cube of the angle θ made by the tangent line (of the linear arch) with the vertical.

To find the depth z by CONSTRUCTION. Having z_0 given, C being the centre of the arch, prolong Ca and make $ab = z_0$; at b draw a ⊥ to Cb, intersecting the vertical through a at some point d; draw the horizontal dc to meet Ca at some point c. Again draw ce ⊥ to Cc meeting ad in e; then $ae = z$, required; a being any point of the linear arch. For, since the similar right triangles involved, we have

$$z_0 = \overline{ab} = \overline{ad}\sin\theta = \overline{ac}\sin\theta.\sin\theta = \overline{ae}\sin\theta\sin\theta\sin\theta$$

$$\therefore\ \overline{ae} = \frac{z_0}{\sin^3\theta}\ ;\ \text{i.e. } \overline{ae} = z. \qquad \text{Q.E.D.}$$

[see (5)]

320. PARABOLA AS LINEAR ARCH. To apply eq. (7) § 318 to a parabola (axis vertical) as linear arch, we must find values of ρ and ρ_0 the radii of curvature at any point and the crown respectively. That is, in the general formula

$$\rho = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \div \frac{d^2y}{dx^2}$$

we must substitute the forms for the first and second differential co-efficients derived from the equation of the curve (parabola) in Fig. 338, i.e. from

$x^2 = 2py$; whence we obtain

$$\frac{dy}{dx}, \text{ or } \cot\theta, = \frac{x}{p} \quad\text{and}\quad \frac{d^2y}{dx^2} = \frac{1}{p}$$

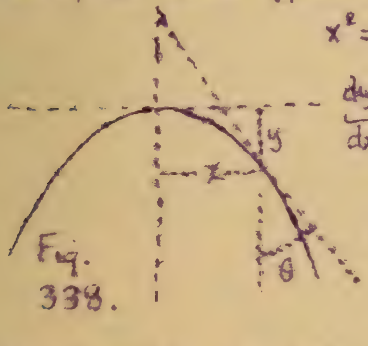


Fig. 338.

Hence

$$\rho = \frac{\left(\sqrt{1+\cot^2\theta}\right)^3}{1 \div p} = p\,\mathrm{cosec}^3\theta$$

i.e. $$\rho = \frac{p}{\sin^3\theta} \quad \text{......} (6)$$

At the vertex $\theta = 90°$

$\therefore\ \rho_0 = p$

Hence by substituting for ρ

and p_0 in eq. (4) of § 318 we obtain

$$z = z_0 = \text{CONSTANT} \quad [\text{Fig. 339}]. \quad (7.)$$

for a parabolic linear arch. Therefore the depth of homogeneous loading must be the same at all points as at the crown; i.e. the load is uniformly distributed along the horizontal. This result might have been anticipated from the fact that a cord assumes the parabolic form when its load (as approximately true for suspension bridges) is uniformly distributed horizontally. (See § 46 in Statics & Dynamics.)

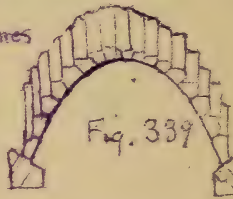


Fig. 339

321. LINEAR ARCH FOR A GIVEN UPPER CONTOUR OF LOADING, the arch itself being the unknown lower contour. Given the upper curve or limit of load and the depth z_0 at crown, required the form of linear arch which will be in equilibrium under the load between itself and that upper curve. In Fig 340 let MON be the given upper contour of load. z_0 is given or assumed. z' and z'' are the respective ordinates of the two curves MN and BAC. Required the equation of BAC.

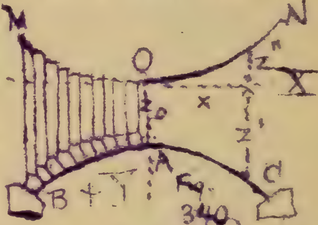


Fig. 340.

As before, the loading is homogeneous so that any portions of it are proportional to the corresponding areas between the curves. (Unity thickness ⊥ to paper.) Now, Fig. 341, regard two consecutive ds's of the linear arch as two links or consecutive blocks bearing at their junction m the load $dP = \gamma(z'+z'')dx$ in which γ denotes the heaviness or weight of a cubic unit of the loading. If T and T′ are the thrusts

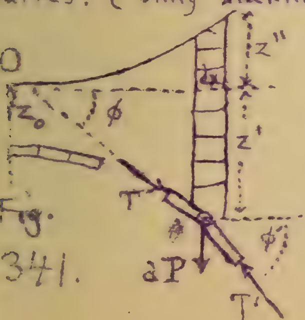


Fig. 341.

exerted on these two blocks by their neighbors (here supposed removed) we have the three forces dP, T, and T', forming a system in equilibrium. Hence from $\Sigma X = 0$

$$T \cos\phi = T' \cos\phi' \quad \ldots\ldots\ldots (1)$$

and

$\Sigma Y = 0$ gives $\quad T' \sin\phi' - T \sin\phi = dP \quad \ldots\ldots (2)$

From (1) it appears that $T\cos\phi$ = constant at all points of the linear arch (just as we found in § 318) and hence = the thrust at the crown, = H, whence we may write

$$T = H \div \cos\phi \quad \text{and} \quad T' = H \div \cos\phi' \qquad (3)$$

Substituting from (3) in (2) we obtain

$$H\,(\tan\phi' - \tan\phi) = dP \quad \ldots\ldots\ldots (4)$$

But $\tan\phi = \dfrac{dz'}{dx}$ and $\tan\phi' = \dfrac{dz' + d^2z'}{dx}$, ($dx$ constant)

while $dP = \gamma(z' + z'')dx$. Hence, putting for convenience $H = \gamma a^2$, (where a = side of an imaginary square of the loading, whose thickness = unity and whose weight = H) we have

$$\frac{d^2z'}{dx^2} = \frac{1}{a^2}(z' + z'') \quad \ldots\ldots \quad (5)$$

as a relation holding good for any point of the linear arch which is to be in equilibrium under the load included between itself and the given curve whose ordinates are z'', Fig. 340.

322. EXAMPLE OF PRECEDING. UPPER CONTOUR A STRAIGHT LINE. Fig. 342. Let the upper contour be a right line and horizontal; then the z'' of eq. 5 becomes zero at all points of ON. Hence drop the prime of z' in eq. 5 and we have

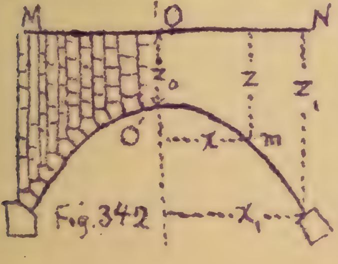


Fig. 342

$$\frac{d^2z}{dx^2} = \frac{z}{a^2}$$

Multiplying which by dy

we obtain $$\frac{dz\,d^2z}{dx^2} = \frac{1}{a^2}\,z\,dz \quad \ldots\ldots \quad (6)$$

This being true of the z, dz, d^2z and dx of each element of the curve $O'B$ whose equation is desired, conceive it written out for each element between O' and <u>any point</u> m, and put the sum of the left-hand members of these equations = to that of the right-hand members, remembering that a^2 and dx^2 are the same for each element. This gives

$$\frac{1}{dx^2}\int_{dz=0}^{dz=dz} dz\,d^2z = \frac{1}{a^2}\int_{z=z_0}^{z=z} z\,dz \;;\; \text{i.e. } \frac{1}{dx^2}\cdot\frac{dz^2}{2} = \frac{1}{a^2}\left[\frac{z^2}{2} - \frac{z_0^2}{2}\right]$$

$$\therefore\; dx = \frac{a\,dz}{\sqrt{z^2 - z_0^2}} = a\,\frac{d\left(\frac{z}{z_0}\right)}{\sqrt{\left(\frac{z}{z_0}\right)^2 - 1}} \quad \ldots\ldots \quad (7.)$$

Integrating (7.) between O' and any point m

$$\Big[x\Big]_0^x = a\left[\log_\varepsilon\left(\frac{z}{z_0} + \sqrt{\left(\frac{z}{z_0}\right)^2 - 1}\right)\right]_{z_0}^{z} \quad \ldots\ldots \quad (8.)$$

$$\text{i.e., } x = a\log_\varepsilon\left[\frac{z}{z_0} + \sqrt{\left(\frac{z}{z_0}\right)^2 - 1}\right] \;;\; \text{or } z = \frac{z_0}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right] \quad (9)$$

This curve is called the TRANSFORMED CATENARY since we may obtain it from a common catenary by altering all the ordinates of the latter in a constant ratio, just as an ellipse may be obtained from a circle. If in eq.(9) a were $= z_0$ the curve would be a common catenary.

Supposing z_0 and the co-ordinates x_1 and z_1 of the point B (abutment) given, we may compute a from eq. 8 by putting $x = x_1$ and $z = z_1$, and solving for a. Then the crown-thrust $H = \gamma a^2$ becomes known, and a can be used in eqs. 8 or 9 to plot points in the curve or linear arch. From eq.(9) we have

$$\left.\begin{array}{l}\text{area}\\ OO'mn\\ \text{Fig. 343}\end{array}\right\} = \int_0^x z\,dx = \frac{z_0}{2}\int_0^x \left[e^{x/a}dx + e^{-x/a}dx\right] = \frac{az_0}{2}\left[e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right] \dots (10)$$

Call this area, A. As for the thrusts at the different joints of the linear arch, see Fig. 343, we have

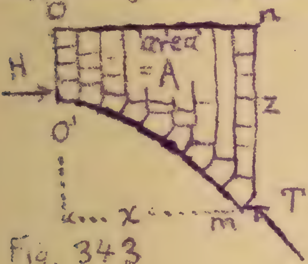


Fig. 343

crown-thrust $= H = \gamma a^2 \dots$ (11)

and at any joint m the thrust

$$T = \sqrt{H^2 + (\gamma A)^2} = \gamma\sqrt{a^4 + A^2} \quad \dots\dots (12)$$

323. REMARKS. The foregoing results may be utilized with arches of finite dimensions by making the arch-ring contain the imaginary linear arch and the joints ⟂ to the curve of the same. Questions of friction and the resistance of the material of the voussoirs are reserved for a succeeding chapter, in which will be advanced a more practical theory dealing with approximate linear arches or "equilibrium polygons" as they will then be called. Still a study of exact linear arches is valuable on many accounts. By inverting the linear arches so far presented we have the forms assumed by flexible and inextensible cords loaded in the same way.

CHAP. VIII. Elements of GRAPHICAL STATICS

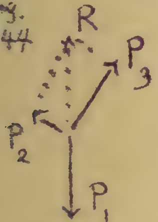
324. DEFINITION. In many respects graphical processes have advantages over the purely analytical, which recommend their use in many problems where celerity is desired without refined accuracy. One of these advantages is that gross errors are more easily detected, and

another that the relations of the forces, distances, etc. are made so apparent to the eye, in the drawing, that the general effect of a given change in the data can readily be predicted at a glance.

GRAPHICAL STATICS is the system of geometrical constructions by which problems in Statics may be solved with the use of drafting instruments, forces as well as distances being represented in amount and direction by lines on the paper, of proper length and position, according to arbitrary scales; so many feet of distance to the linear inch of paper, for example, for distances; and so many pounds or tons to the linear inch of paper for forces. Of course results should be interpreted by the same scale as that used for the data. The parallelogram of forces is the basis of all constructions for combining and resolving forces.

325. FORCE POLYGONS AND CONCURRENT FORCES IN A PLANE. If a material point is in equilibrium under three forces P_1 P_2 and P_3 (in the same plane of course) Fig. 344, any one of them, as P_1, must be equal and opposite to R the resultant of the other two (diagonal of their parallelogram). If now we lay off to some convenient scale a line in Fig. 345 to represent P_1 and ‖ to P_1 in Fig. 344; and then from the pointed end of P_1 a line equal and ‖ to P_2 and laid off pointing the same way, we note that the line remaining to close the triangle in Fig. 345 must be = and ‖ to P_3, since that triangle is nothing more than the left-hand half-parallelogram of Fig. 344. Also, in 345, to close the triangle properly the directions of the arrows must be continuous POINT to BUTT.

Fig. 344



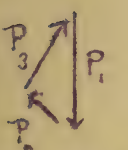


Fig. 345

round the periphery. Fig. 345 is called a FORCE POLYGON; of three sides only in this case. By means of it, given any two of the three forces which hold the point in equilibrium, the third can be found, being equal and ∥ to the side necessary to "close" the force polygon.

Similarly, if a number of forces in a plane hold a material point in equilibrium, Fig. 346, their force polygon, Fig. 347, must close, whatever be the order in which its sides are drawn. It is more convenient however, to follow the order of the forces around the point O. If P_5 e.g., is the unknown force which is to balance the other four (i.e., is their anti-resultant), we draw the sides of the force polygon from A round to B; then the line BA represents the required P_5 in amount and direction, since the arrow BA must follow the continuity of the others (point to butt), Fig. 347.

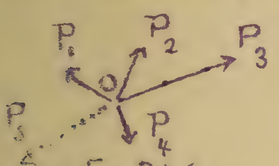


Fig. 346

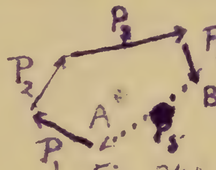
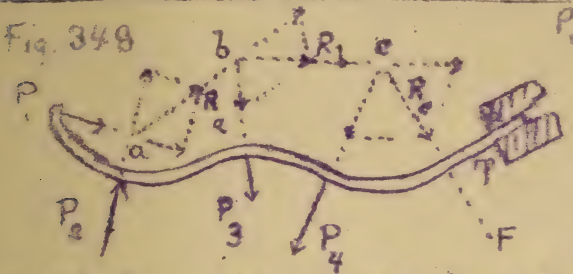


Fig. 347

If the arrow BA were pointed at the extremity B, then it gives, obviously, the amount and direction of the resultant of the four forces $P_1 \ldots P_4$. The foregoing shows that if a system of CONCURRENT FORCES IN A PLANE is in equilibrium, its force polygon must close.

326. NON-CONCURRENT FORCES IN A PLANE. Given a system of non-concurrent forces in a plane, acting on a rigid body, required graphic means of finding their resultant, and anti-resultant; also of expressing conditions of equilibrium. The resultant must be found in amount and direction; and also in position (i.e. its line of action must be determined). E.g., Fig. 348 shows a curved rigid beam fixed in a vise at T, and also under the action of forces P_1 P_2 P_3 and P_4 (besides the action of the vise); required the resultant of

Fig. 348



P_1, P_2, P_3 and P_4. By the ordinary parallelogram of forces we combine P_1 and P_2 at a, the intersection of their lines of action, into a resultant R_a; then R_a with P_3 at b, to form R_b; and finally R_b with P_4 at c to form R_c which is ∴ the resultant desired, i.e. of P_1 ..... P_4, and cF is its line of action. The separate force triangles (half-parallelograms) by which the successive partial resultants R_a etc., were obtained are again drawn in Fig. 349. Now since R_c, acting in the line c ... F ([illegible]), is the resultant of P_1 .... P_4, it is plain that a force R'_c equal to R_c and acting along c ... F but in the OPPOSITE direction would balance P_1 ... P_4. That is, the forces P_1 P_2 P_3 P_4 and R'_c would form a system in equilibrium. The force R'_c, then, represents the action of the vise at T upon the beam. Hence replace the vise by the force R'_c acting in the line F... c, to do which requires us to imagine a rigid prolongation of that end of the beam to intersect F.... c. This is shown in Fig. 350 where the whole beam is FREE, in equilibrium, and is in precisely the same state of stress, part for part, as in Fig. 348. Also, by combining in one FORCE DIAGRAM in Fig. 351, all the force triangles of Fig. 349 (by making their common sides co-incide, and putting R'_c instead of R_c), and making dotted all forces other than those

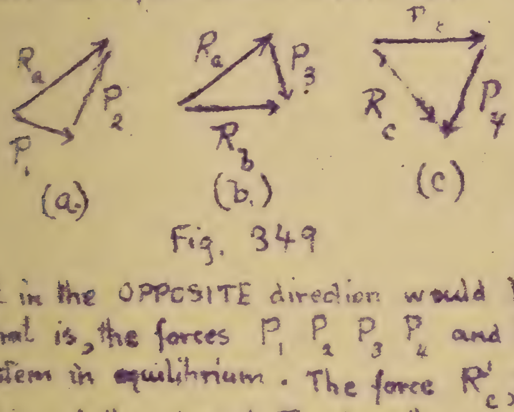


Fig. 349

of Fig 350) we have a figure to be interpreted in connection with Fig. 350.

Fig. 350

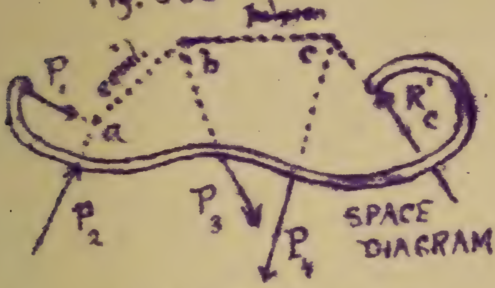
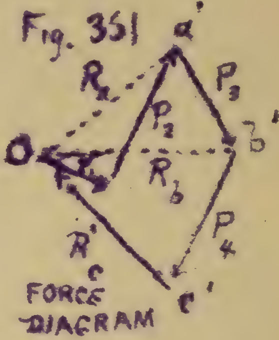


Fig. 351



Here we note, first, that in the figure called a Force Diagram, P_1 P_2 P_3 P_4 and R'_c form a closed polygon and that their arrows follow the same direction, point to butt, around the perimeter; which proves that one condition of equilibrium for a system of non-concurrent forces in a plane is that its force polygon (O a' b' c') must close. Next note that ab is ‖ to Oa', and bc ‖ to O'b'; hence if the force diagram has been drawn (including the RAYS, as they are called, Oa' and Ob'), in order to determine the amount and direction of R'_c or any other one force, we may find the line of action of R'_c in the SPACE DIAGRAM as we may call the figure showing to a true scale the form of the rigid body and the lines of action of the forces, by drawing from a the intersection of P_1 and P_2 a line ab ‖ to Oa' to intersect P_3 at some point b; then bc ‖ to Ob' to intersect P_4 at c; then cF ‖ to Oc' will be the required line of action of R'_c the anti-resultant of P_1 P_2 P_3 and P_4.

abc is called an EQUILIBRIUM POLYGON. This one has but two segments ab and bc (Still, the lines of action of P_1 and R'_c may also be considered as segments)

The segments of the equilibrium polygon are ‖ to the respective rays of the force diagram.

Hence for the equilibrium of a system of NON-CONCURRENT FORCES in a plane it is necessary not only that its force polygon must close, but also that the first and last segments of the corresponding equilibrium polygon shall co-incide with the resultants of the first two forces and of the last two forces, respectively, of the system. E.g. ab co-incides with the line of action of the resultant of P_1 and P_2; bc with that of P_4 and R'_c. Evidently the equil. polygon will be different with each different order of forces in the force polygon, or different choice of a POLE, O. But if the order of forces be taken as above, as they occur along the beam, or structure, and the pole taken at the "butt" of the first force in the force polygon, there will be only one; (and this one will be called the SPECIAL EQUILIBRIUM POLYGON" in the chapter on arch-ribs, and the "true linear arch" in dealing with the stone arch. After the RAYS (dotted in Fig. 351) have been added, by joining the pole to each vertex with which it is not already connected the final figure may be called the force diagram.

It may sometimes be convenient to give the name of rays to the two forces of the force polygon which meet at the pole, in which case the first and last segments of the corresponding equil. polygon will coincide with the lines of action of those forces in the SPACE-DIAGRAM (as we may call the representation of the body or structure on which the forces act. This "space diagram" shows the real field of action of the forces, while the force-diagram, which may be placed in any convenient position on the paper, shows the magnitudes and directions of the forces acting in the former diagram, its lines being interpreted on a scale of so many lbs. or tons to the inch of paper; in the space-diagram we deal with a scale of so many feet to the inch of paper

We have just found that if any vertex or corner of the

closed force polygon be taken as a pole, and rays drawn from it to all the other corners of the polygon, and a corresponding equil. polygon drawn in the space diagram, the first and last segments of the latter polygon will coincide with the first and last forces according to the order adopted (or with the resultants of the first two and last two, if more convenient to classify them thus). It remains to utilize this principle.

327. LOADED ROD LEANING AGAINST A SMOOTH WALL. The graphic relations just found may be made the means of solving problems involving non-concurrent forces in a plane, when there are most three unknown quantities. A case is now taken for illustration. Fig. 352 shows a rod hinged at O, leaning against a smooth vertical wall at B, and bearing two loads. Required the (horizontal) reaction H' of the wall, and the horizontal and vertical components $\dots$, V_0 and H, of the pressure at the hinge, which is unknown in direction and hence best replaced by its components in known directions.

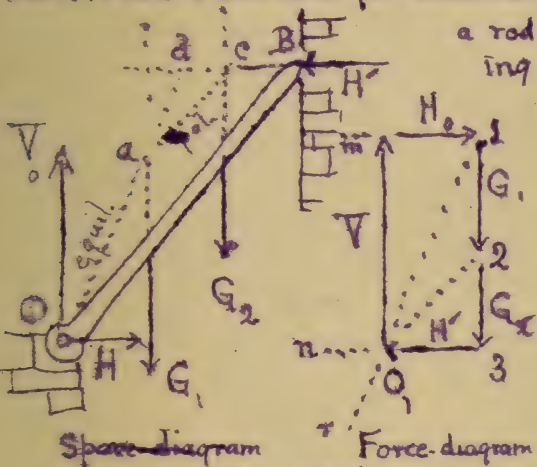


Fig. 352.

We first suppose the problem solved, graphically, so as to judge of the best method of determining the unknown from the known parts. The force polygon (full lines in force diagram) consists of lines laid off end to end, and respectively equal and ∥ (to scale) to the forces of the system, in the order V_0, H_0, G_1, G_2, and H'.

For equilibrium this force polygon must close, i.e., the

last force, H', must terminate in O_1, the butt of the first force V_0, and the arrows must have a continuous series point to butt, around the polygon. For this closing to take place it is evident that H_0 and H' must be equal in amount and opposite in direction; also that V_0 is equal in amount to $G_1 + G_2$, and opposite to them in direction. [Corresponding to the analytical conditions $\Sigma X = 0$ and $\Sigma Y = 0$]

Now connect O_1, as a pole, with all the vertices of the force-polygon with which it is not already connected; i.e., draw the RAYS $O_1 \ldots 1$, $O_1 \ldots 2$, only two in this case, though the forces V_0 and H_0, meeting at O_1, may also be considered as rays. Also draw from O in the space-diagram, the line Oa ∥ to the ray $O_1 \ldots 1$ in the force diagram, and we have the first segment of the equilibrium polygon, a being its intersection with the line of action of G_1, which is the force in the force diagram to whose butt the ray $O_1 \ldots 1$ has been drawn. Then through a, draw a line ac ∥ to the second ray $O_1 \ldots 2$ of the force diagram to intersect G_2, the force to whose butt the second ray has been drawn. This second segment of the equil. polygon should strike c, the intersection of G_1 and H' in the space-diagram (if not, the system is not balanced)

The length of the first ray $O_1 \ldots 1$ gives the amount of the resultant of V_0 and H_0, while Oa in the space diagram is the line of action of that resultant. The length of the second ray $O_1 \ldots 2$ gives the amount of the resultant of V_0 H_0 and G_1; this may be called the second partial resultant and has ac for its line of action; and so on for more forces. The last partial resultant must balance the resultant of the one or two remaining forces. In this instance $O_1 \ldots 2$, the re-

sultant of V_o H_o and G_1 acts along ac and balances the resultant of H' and G_2 acting at c; or, if we consider $O_1 \dots 3$ as a ray, we may say that $O_1 \dots 3$ (pointing toward the right) the resultant of V_o H_o, G_1 and G_2 acts along cB and balances H', and then we would consider cB as the last segment of the equil. polygon.

It is evident, then, that the segments of the equil. polygon are the lines of action of the successive partial resultants, whose magnitudes must be taken from the corresponding parallel rays of the force diagram.

We now proceed to find out how the unknown parts may be constructed from the known, the latter being G_1 and G_2 (both in amount and position) and the lines of action of H' H_o and V_o. We can therefore, at the outset, lay off only the two sides G_1 and G_2 of the force polygon in the force diagram and indefinite horizontal lines $1 \dots m$ and $1 \dots n$ in the lines of H_o and H'. How shall we determine O_1, the pole?

If we consider that the resultant of G_1 and G_2 must balance that of V_o H_o and H'; in other words that the result of combining H' with the resultant of V_o and H_o will be a force passing through the point d and equal and opposite to the resultant of G_1 and G_2; it is evident that the resultant of G_1 and G_2 must intersect H' at d and thus determine dO the direction of aO the first segment of the equil. polygon. The line of action of this resultant (of G_1 and G_2) is most conveniently found by the next paragraph (328), and the line of action of H' is known. Hence d is determined by their intersection. Then by drawing through the point 1 of the force diagram a line ∥ to dO, we determine the first ray and the POLE, O_1, which is the intersection of $1 \dots r$ with $3 \dots n$. With O_1 now determined, it is a simple matter to fill out the

force diagram by drawing V_o and H_o ∥ to their respective directions, and the ray $O_1 \ldots 2$. The equil. polygon is also quickly finished in a manner previously explained. Then the three unknowns V_o H_o and H' become known graphically, and may be scaled off according (to the force-scale adopted) in the force diagram.

328. TO FIND THE RESULTANT OF SEVERAL FORCES IN A PLANE. This might be done as in § 326, but since frequently a given set of forces are parallel, or nearly so, a special method will now be given, of great convenience in such cases. Fig. 353. Let P_1 P_2 and P_3 be the given forces whose resultant is required. Let us first find their

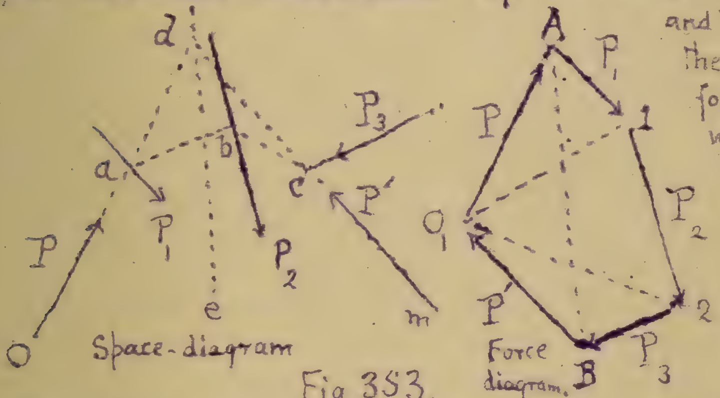


Fig. 353.

anti-resultant, or force which will balance them. This anti-resultant may be conceived as decomposed into two components P and P′ one of which, say P, is arbitrary in amount and position. Assuming P, then, at convenience, it is required to find P′. The five forces must form a balanced system, hence if beginning at O_1 we lay off a line $O_1A = P$ by scale, the $A1 = P_1$, and so on (point to butt), the line BO_1 necessary to close the force polygon is = P′ required. Now form the corresponding equil. polygon in the space diagram in the usual way, viz: through a the intersection of P and P_1 draw ab ∥ to the ray $O_1 \ldots 1$

(which connects the pole O_1 with the . of the last force mentioned). From b, where ab intersects the line of P_2, draw bc, ∥ to the ray O_1...2, till it intersects the line of P_3. A line mc drawn through c and ∥ to the P' of the force diagram is the line of action of P'.

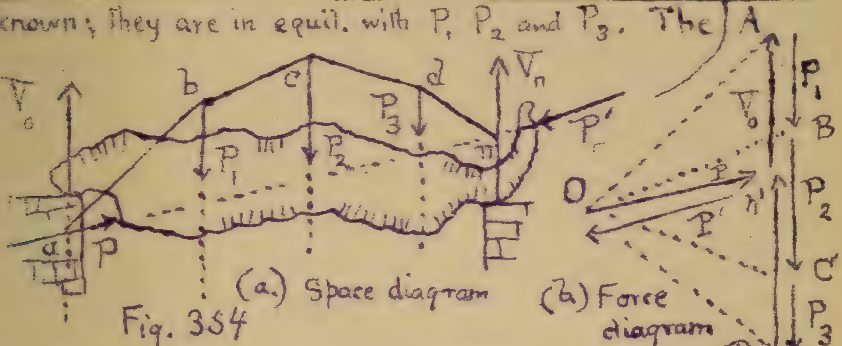
Now the resultant of P and P' is the anti-resultant of P_1 P_2 and P_3; ∴ d, the intersection of the lines of P and P' is a point in the line of action of the anti-resultant required, while its direction and magnitude are given by the line BA in the force diagram; for BA forms a closed polygon both with P_1 P_2 P_3, and with P P'. Hence a line through d ∥ to BA, viz. de, is the line of action of the anti-resultant (and hence of the resultant) of P_1 P_2 P_3.

Since in this construction P is arbitrary, we may first choose O_1, arbitrarily, in a convenient position, i.e., in such a position that by inspection the segments of the resulting equil-polygon shall give fair intersections and not pass off the paper. If the given forces are parallel the device of introducing the oblique P and P' is quite necessary.

The result of this construction may be stated as follows, (regarding Oa and cm as segments of the equil. polygon as well as ab and bc): IF ANY TWO SEGMENTS OF AN EQUIL. POLYGON BE PROLONGED, their intersection is a point in the line of action of the RESULTANT OF THOSE FORCES acting at the vertices intervening between the given segments. Here, the resultant of P_1 P_2 P_3 acts through d.

329. VERTICAL REACTIONS OF PIERS, etc. Fig 354. Given the vertical forces or loads P_1 P_2 and P_3 acting on a rigid body (beam, or truss) which is supported by two piers having smooth horizontal surfaces (so that the reactions must be vertical), required the reactions V_0 and V_n of the piers. For an instant suppose V_0 and V_n

known; they are in equil. with P_1, P_2 and P_3. The



(a.) Space diagram (b.) Force diagram

Fig. 354

introduction of the equal and opposite forces P and P' in the same line will not disturb the equilibrium. Taking the seven forces in the order P V_0 P_1 P_2 P_3 V_n and P', a force polygon formed with them will close (see (b.) in fig. where the forces which really lie on the same line are slightly separated). With O, the butt of P, as a pole draw the rays of the force diagram OA, OB, etc. The corresponding equil. polygon begins at a, the intersection of P and V_0 in the space diagram, and ends at n the intersection of P' and V_n. Join an. Now since P and P' act in the same line, an must be that line and must be ∥ to P and P' of the force diagram. Since the amount and direction of P and P' are arbitrary, the position of the pole O is arbitrary, while P_1, P_2, and P_3 are the only forces known in advance in the force diagram.

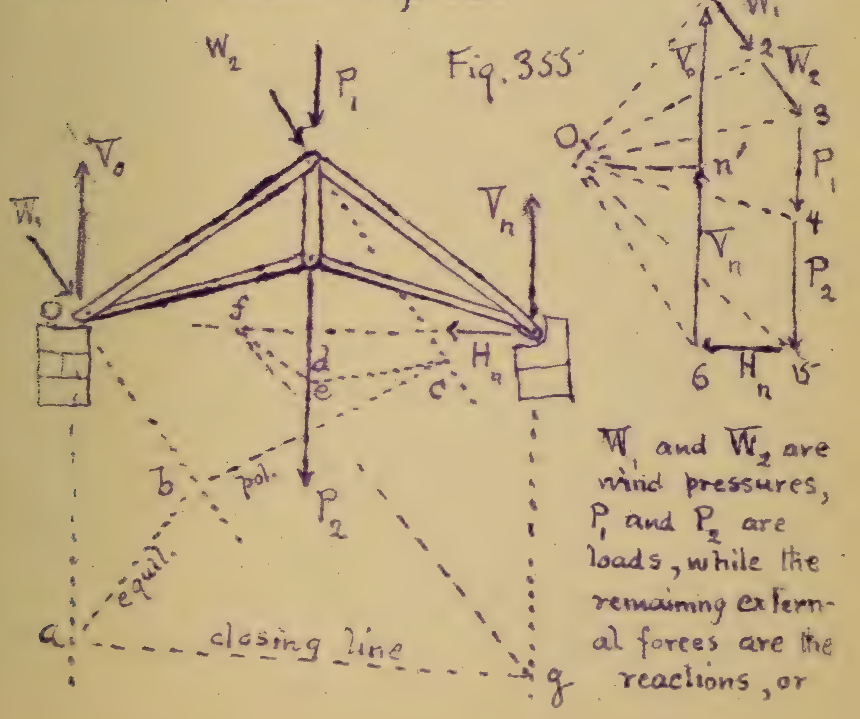
Hence V_0 and V_n may be determined as follows: lay off the given loads P_1, P_2, etc. in the order of their occurrence in the space diagram, to form a "load-line" AD (see (b.) fig. 354) as a beginning for a force-diagram; take any convenient pole O, draw the rays OA, OB, OC and OD. Then beginning at any convenient point a in the vertical line containing the unknown V_0, draw ab ∥ to OA, bc ∥ to OB, and so on, until the last segment (dn in this case) cuts the vertical containing the unknown V_n

in some point n. Join an (this is sometimes called a *closing line*) and draw a ‖ to it through O, in the force-diagram. This last line will cut the "load-line" in some point n', and divide it in two parts, which are respectively V_o and V_n required.

COROLLARY. Evidently for a given system of loads, in given vertical lines of action, and for two given piers, or abutments, having smooth horizontal surfaces, the location of the point n' on the load line is INDEPENDENT OF THE CHOICE OF A POLE.

Of course, in treating the stresses and deflection of the rigid body concerned, Pons P are left out of account, as being imaginary and serving only a temporary purpose.

330. APPLICATION OF FOREGOING PRINCIPLES TO A ROOF TRUSS. Fig. 355



W_1 and W_2 are wind pressures, P_1 and P_2 are loads, while the remaining external forces are the reactions, or

supporting forces, V_o V_n and H_n, may be found by preceding §§. (We here suppose that the right abutment furnishes all the horizontal resistance; none at the left.)

Lay off the forces (known) W_1 W_2 P_1 and P_2 in the usual way, to form a portion of the closed force polygon. To close the polygon it is evident we need only draw a horizontal through 5 and limit it by a vertical through 1. This determines H_n but it remains to determine n' the point of division between V_o and V_n. Select a convenient pole O_1, and draw rays from it to 1, 2, etc. Assume a convenient point a in the line of V_o in the space diagram, and through it draw a line ∥ to $O_1 1$ to meet the line of W_1 in some point b; then " " " $O_1 2$ " "
" " " W_2 " " " c; " thro' c " " $O_1 3$ " "
" " " P^2 " " " d; " " d " " $O_1 4$ " "
" " " P^1 " " " e, (e is identical with d, since P_1 and P_2 are in the same line); then ef ∥ to $O_1 5$ to meet H_n in some point f; " fg ∥ to $O_1 6$
" " V_n " " " g.

Now join ag, the "closing-line", and draw a ∥ to it through O_1 to determine n', the required point of division between V_o and V_n on the vertical 16. Hence V_o and V_n are now determined as well as H_n.

The use of the arbitrary pole O_1 implies the temporary employment of a pair of opp. and equal forces in the line ag, the amount of either being $= O_1 n'$.

Having now all the external forces acting on the truss we proceed to find the pulls and thrusts in the individual pieces, on the following plan. The truss being pin-connected, no piece extending beyond a joint, and all loads being considered to act at joints, the action, pull or thrust, of each piece on the joint at either extremity will be

in the direction of the piece, i.e., in a known direction, and the pin of each joint is in equilibrium under a system of concurrent forces consisting of the loads (if any) at the joint and the pulls or thrusts exerted upon it by the pieces meeting there. Hence we may apply the principles of § 325 to each joint in turn. See Fig. 356

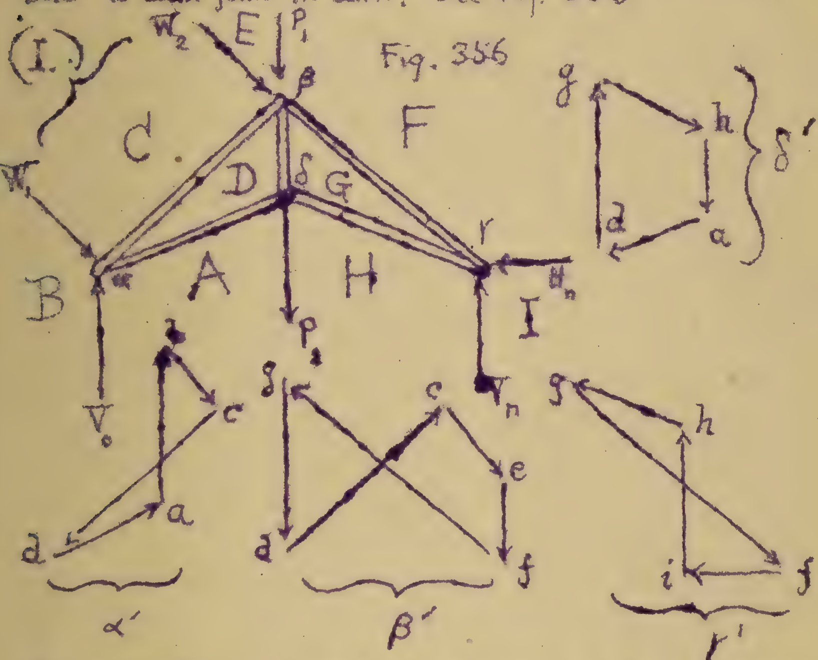


Fig. 356

In constructing and interpreting the various force polygons, R.H. Bow's convenient notation will be used; this is as follows. In the space diagram a capital letter [A B C etc.] in each triangular cell of the truss, and also in each angular space in the outside outline of the truss between the external forces and the adjacent truss-pieces. In this way we can speak of the force W_1 as the force BC, of W_2 as the force CE, the stress in the piece $\alpha\beta$ as the force CD, and so on. That is, the stress in any one piece can be named from the letters in the spaces bordering its two sides. Corresponding

To these capital letters in the spaces of the space-diagram, small letters will be used at the vertices of the closed force-polygons (one polygon for each joint) in such a way that the stress in the piece CD, for example, shall be the force dc of the force polygon belonging to any joint in which that piece terminates; the stress in the piece FG by the force fg in the proper force polygon, and so on.

At (I.) in Fig. 356 the whole truss is shown free, in equilibrium under the external forces. To find the pulls or thrusts (i.e. tensions or compressions) in the pieces, consider that if all but two of the forces of a closed force polygon are known in magnitude and direction, while the directions, only, of those two are known, the whole force polygon may be drawn, thus determining the amounts of those two forces by the lengths of the corresponding sides.

We must ∴ begin with a joint where no more than two pieces meet, as at α, [call the joints α β γ δ, and the corresponding force polygons α' β' etc. Fig. 356.] Hence at α' (anywhere on the paper) make ab ∥ and = (by scale) to the known force AB (i.e. V_0) pointing it at the upper end, and from this end draw bc = and ∥ to the known force BC (i.e. W_1) pointing this at the lower end. To close the polygon draw through c a ∥ to the piece CD, and through a a ∥ to AD; their intersection determines d, and the polygon is closed. Since the arrows must be point to butt round the periphery, the force with which the piece CD acts on the pin of the joint α is a force of an amount = cd and in a direction from c toward d; hence the piece CD is *in compression*; whereas the action of the piece DA upon the pin at α is from d toward a (direction of arrow) and hence DA is *in tension*. Notice that in constructing the force polygon α' a right-handed (or clock-wise) relation has been observ-

ed in considering in turn the spaces A B C and D, round the joint α. A similar order will be found convenient in each of the other joints.

Knowing now the stress in the ~~piece~~ CD, (as well as in DA) all but two of the forces acting on the pin at the joint β are known and accordingly we begin a force polygon, β', for that joint by drawing dc = and ∥ to the dc of polygon α', *but pointed in the opposite direction*, since the action of CD on the joint β is equal and opposite to its action on the joint α (this disregards the weight of the piece). Through c draw ce = and ∥ to the force CE (i.e., W_2) and pointing the same way; then ef = and ∥ to the ~~last~~ EF (i.e. P_1) and pointing downward. Through f draw a ∥ to the piece FG and through d a ∥ to the piece GD, and the polygon is closed, thus determining the stresses in the pieces FG and GD. Noting the pointing of the arrows, we readily see that FG is in compression while CD is in tension.

Next pass to the joint δ, and construct the polygon δ', thus determining the stress gh in GH and that ad in AD; this last force ad should check with its equal and opposite ad already determined in polygon α'. Another check consists in the ~~proper~~ closing of the polygon γ' all of whose sides are now known.

A compound stress-diagram may be formed by superposing the polygons already found in such a way as to make equal sides ~~co-incide~~; but the character of each stress is not so readily perceived then as when they are kept separate.

In a similar manner we may find the stresses in any pin-connected frame-work (in one plane and having no redundant pieces) under given loads, provided all the supporting forces or reaction can be found. In the case of a braced-arch (truss) as shown in Fig. 357, hinged to the

abutments at both ends and not free to slide laterally upon them, the reactions at O and B depend, in amount and direction, not only upon the equations of Statics, but on the form and elasticity of the arch-truss. Such cases will be treated later under arch-ribs, or curved beams.

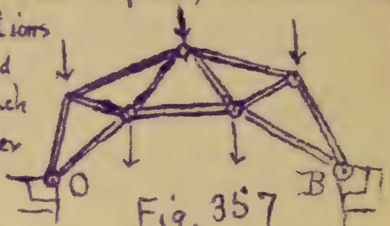


Fig. 357

332. THE SPECIAL EQUIL. POLYGON. ITS RELATION TO THE STRESSES IN THE RIGID BODY.

Reproducing Figs. 350 and 351 in Figs. 358 and 359,

Fig. 358

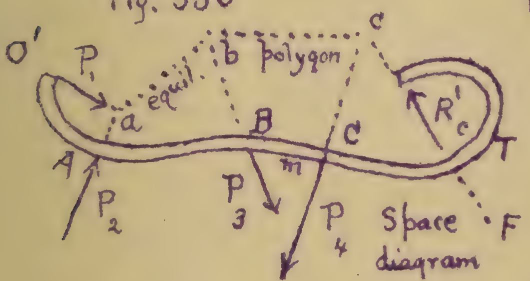
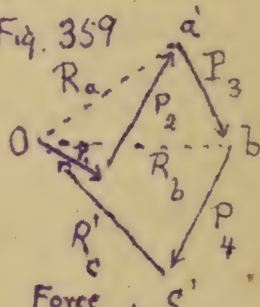


Fig. 359



(where a rigid curved beam is in equilibrium under the forces P_1 P_2 P_3 P_4 and R'_c) we call a..b...c the *special equil. polygon* because it corresponds to a force diagram in which the same order of forces has been observed as that in which they occur along the beam (from left to right here). From the relations between the force diagram and equil. polygon, this SPECIAL equil. polygon in the space diagram has the following properties in connection with the corresponding rays (dotted lines) in the force diagram.

The stresses in any cross-section of the portion O'A of the beam, are due to P_1 alone; those of any cross section on AB to P_1 and P_2, i.e. to *their resultant* R_a, whose magnitude is given by the line Oa' in the force dia-

gram, while its line of action is ab the first segment of the equil. polygon. Similarly, the stresses in BC are due to P_1 P_2 and P_3; i.e., to their resultant R_b acting along the segment bc, its magnitude being $= O'b'$ in the force diagram. E.g. If the section at m be exposed, considering $O'AB_m$ as a free body we have (see Fig. 360)

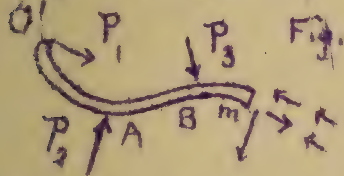


Fig. 360

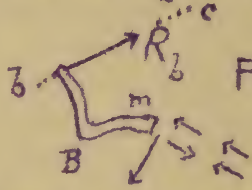


Fig. 361

the elastic stresses (or internal forces) at m balancing the exterior "applied forces" P_1 P_2 and P_3. Obviously, then, the stresses at m are just the same as if R_b the resultant of P_1 P_2 and P_3, acted upon an imaginary rigid prolongation of the beam intersecting bc (see Fig. 361). R_b might be called the <u>anti-stress-resultant</u> for the portion BC of the beam. We may ∴ state the following:

If a rigid body is in equilibrium under a system of NON-CONCURRENT FORCES in a plane, and the special equilibrium polygon has been drawn, then each ray of the force diagram is the anti-stress-resultant of that [illegible] *portion of the beam which corresponds to the segment of the equilibrium polygon to which the ray is parallel; and its line of action is the segment just mentioned.*

Evidently if the body is not one rigid piece, but composed of a ring of uncemented blocks (or voussoirs), it may be considered rigid only so long as no slipping takes place or disarrangement of the blocks; and this requires that the "anti-stress-resultant for a given joint between two blocks shall not lie outside the bearing surface of the joint, nor make too small an angle with it lest slipping or tipping occur.

For an example of this see Fig. 362, showing a line of three blocks in equilibrium under five forces. The pressure borne at the joint MN is $= R_a$ in the force-diagram and

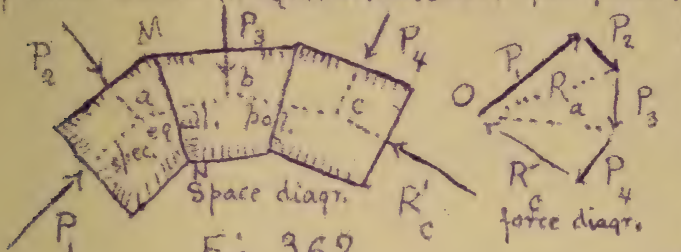


Fig. 362

acts in the line ab. This construction supposes all the forces given except one, in amount and position, and that that one could easily be found in amount as being the side remaining to close the force polygon, while its position would depend on the equil. polygon. But in practice the Two forces P_1 and R'_c are generally unknown, hence the point O, or pole of the force diagram, can not be fixed nor the special equil. polygon located, until other considerations, outside of those so far presented, are brought into play. In the progress of a problem, as will be seen, it will be necessary to use arbitrary trial positions for the pole O, and corresponding trial equil. polygons.

CHAP. IX.

Graphical Statics of Vertical Forces.

333. REMARKS. In problems to be treated subsequently (either the stiff arch-rib, or the block-work of an arch-ring, of masonry) when the body is considered free all the forces holding it in equil. will be vertical (loads, due to gravity) except the reactions at the two extremities, thus, Fig. 363; but for convenience each reaction will be replaced by its horizontal and vertical components (see Fig. 364) The two H's are of course equal, since

they are the only forces in the system. Henceforth all equil. polygons under discussion will be understood to imply this kind of system of forces. P_1 P_2 etc. will represent the "loads"; V_0 and V_n the vertical components

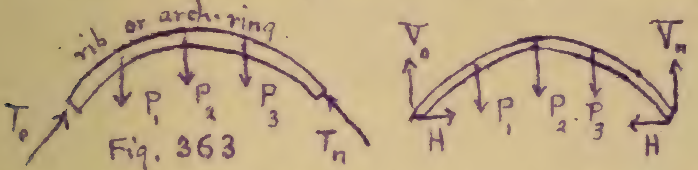
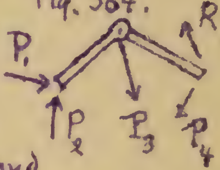


Fig. 363

of the abutment reactions; H the value of either horizontal component of the same.

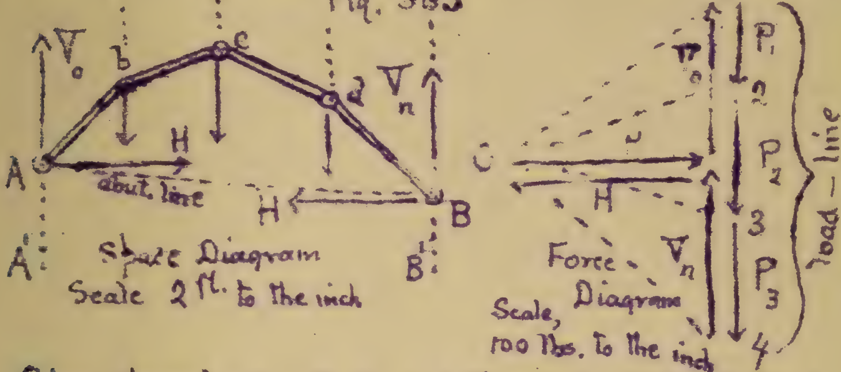
334. CONCRETE CONCEPTION OF AN EQUIL. POLYGON. Any equil. polygon has this property, due to its mode of construction, viz.: If the *ab* and *bc* of Fig 358 were imponderable straight rods, jointed at *b* without friction, they would be in equilibrium under the system of forces there given; see Fig. 364. The rod *ab* suffers a compression equal to the R_a of the force diagram, Fig. 359, and *bc* a compression $= R_b$. In some cases these rods might be in tension, and would then form a set of links playing the part of a suspension-bridge cable; see § 44.

Fig. 364.



335. EXAMPLE OF EQUIL. POLYGON DRAWN TO VERTICAL LOADS. Fig. 365 [The structure bearing the given loads is not shown, but simply the imaginary rods, or segments of an equil. polygon, which would support the given loads in equilibrium if the abutment points A and B, to which the terminal rods are hinged, were firm. In the present case this equilibrium is unstable since the rods form a standing structure; but if they were hanging, the equilibrium would be stable. Still, in the present case, a very light bracing, or a little friction at all joints would make the equilibrium stable.

Fig. 365



Given three loads P_1 P_2 and P_3, and two "abutment verticals" A' and B' in which we desire the equil. polygon to terminate, lay off as a "load-line", to scale, P_1 P_2 and P_3 end to end in their order. Then selecting any pole, O, draw the rays O1, O2, etc. of a force diagram (the V's and P's, though really on the same vertical, are separated slightly for distinctness; also the H's, which both pass through O and divide the load-line into V_0 and V_n) We determine a corresponding equil. polygon by drawing through A (any point in A') a line ∥ to O...1 to intersect P_1 in some point b; through a ∥ to O...2 and so on, until B' the other abutment-vertical is struck in some point B. AB is the "abutment-line" or closing-line.

By choosing another point for O, another equil. polygon would result. As to which of the infinite number (which could thus be drawn, for the given loads and the A' and B' verticals) is the special equil. polygon for the arch-rib or stone arch, or other structure, on which the loads rest, is to be considered hereafter. In any of the above equil. polygons the imaginary series of jointed rods would be in equilibrium.

336. USEFUL PROPERTY of an EQUIL. POL. FOR VERTICAL LOADS. (Particular case of § 328) See

Fig. 366. In any equil. polygon, supporting vertical loads, consider as free any number of consecutive segments, or rods, with the loads at their joints, e.g. the 5th and 6th, and portions of the 4th and 7th, which we suppose cut and the compressive forces in them put in, T_4 and T_7, in order to consider to consider 4 5 6 7 as a free body.

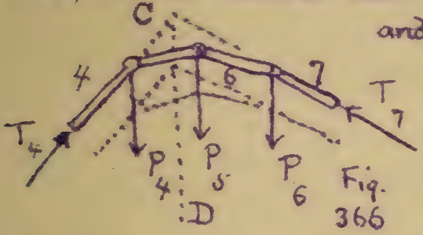


Fig. 366

For equil., according to Statics, the lines of action of T_4 and T_7 (the compressions in those rods) must intersect in a point in the line of action of the resultant of P_4 P_5 and P_6, i.e. of the loads occurring at the intervening vertices. That is the point C must lie in the vertical containing the centre of gravity of those loads. Since the position of this vertical must be independent of the particular equil. polygon used, any other (dotted lines in fig. 366) for the same loads will give the same results. Hence the vertical CD, containing the centre of gravity of any number of consecutive loads, is easily found by drawing the equil. polygon corresponding to any convenient force diagram having the proper load-line.

USEFUL RELATIONS BETWEEN FORCE DIAGRAMS AND EQUILIBR. POLYGONS (for vertical loads)

337. RESUMÉ OF CONSTRUCTION. Fig. 367. Given the loads P_1, etc. Their verticals, and the two abutment verticals A' and B', in which the abutments are to lie; we lay off a load-line 1...... 4, take any convenient pole, O, for a force-diagram and complete the latter. For a corresponding equil. polygon, assume any point A in the vertical A', for an abutment, and draw the successive segments Ab, bc, etc. respectively parallel to

the inclined lines of the force diagram (RAYS), thus determining finally the abutment B, in B', which (B) will not in general lie in the horizontal through A.

Now join AB, calling AB the ~~abutment-line~~, and a ∥ to it through O, thus fixing the point n' on the load-line. This point n', as above determined, is INDEPENDENT OF THE LOCATION OF THE POLE, O, (proved in § 329) and divides the load-line into two portions ($V_0' = 1 \ldots n'$, and $V_n' = n' \ldots 4$) which are the vertical pressures which two supports in the verticals A' and B' would sustain if the given loads rested on a horizontal rigid bar as in Fig. 368.

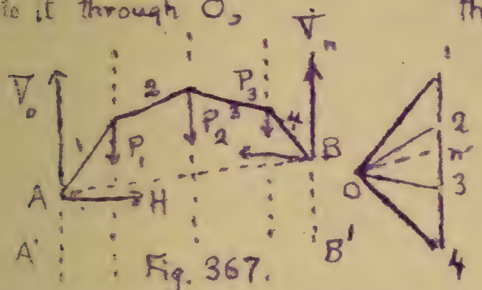


Fig. 367.

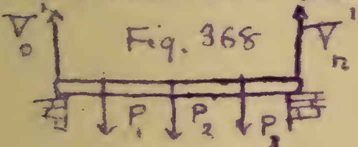


Fig. 368

See § 329. Hence to find the point n' we may use any convenient pole O [N.B. The forces V_0 and V_n of Fig. 367 are not identical with V_0' and V_n', but may be obtained by dropping a ⊥ from O to the load line, thus dividing the load-line into two portions which are V_0 (upper portion) and V_n.

338. THEOREM. *The vertical dimensions of any two equilibrium polygons, drawn to the same loads, load-verticals, and abutment verticals, are inversely proportional to their H's (or pole-distances).* We here regard an equil. polygon and its abutment-line as a closed figure. Thus, in Fig. 369, we have two force-diagrams (with a common load-line, for convenience) and their corresponding equil. polygons, for the same loads and verticals. From § 337 we know that On' is ∥ to AB and O_0n' is ∥ to A_0B_0. Let CD be any vertical cutting the

first segments of the two equil. polygons. Denote the intercepts thus determined by z' and z'_0, respectively.

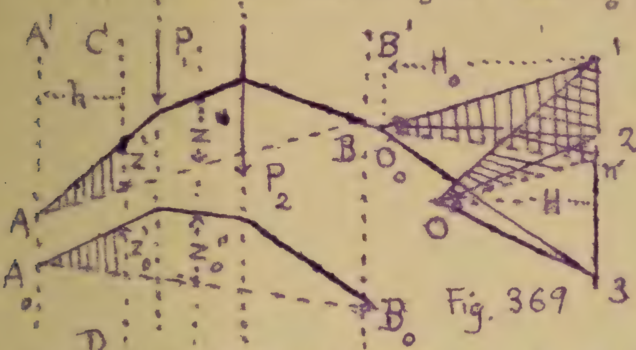


Fig. 369

From the parallelisms just mentioned, and others more familiar, we have the $\Delta\ O1n'$ similar to the $\Delta\ Az'$ (shaded) and the $\Delta\ O_01n'$ similar to $\Delta\ A_0z'_0$. Hence the proportions between bases and altitudes

$$\left\{ \frac{\overline{1n'}}{H} = \frac{z'}{h} \quad \text{and} \quad \frac{\overline{1n'}}{H_0} = \frac{z'_0}{h} \right\}$$

$\therefore\ z' : z'_0 :: H_0 : H$ The same kind of proof may easily be applied to the vertical intercepts in any other segments, e.g. z'' and z''_0. Q.E.D.

339. COROLLARIES to the foregoing. It is evident that:

(1.) If the pole of the force-diagram be moved along a vertical line, the equil. polygon changing its form in a corresponding manner, the vertical dimensions of the equil. polygon remain unchanged; and

(2) If the pole move along a straight line which contains the point n', the direction of the abutment-line remains constantly parallel to the former line, while the vertical dimensions of the equilibrium polygon change in inverse proportion to the *pole distance*, or H, of the force diagram. [H is the ⊥ distance of the pole from the load-line, and is called the *pole-distance*.]

§ 340. LINEAR ARCH AS EQUILIBRIUM POLYGON. (See § 316) If the given loads are infin-

itely small with infinitely small horizontal spaces between them, any equilibrium polygon becomes a LINEAR ARCH. Graphically we can not deal with these infinitely small loads and spaces, but from § 336 it is evident that if we replace them, in successive groups, by finite forces, each of which = the sum of those composing one group and is applied through the centre of gravity of that group, we can ~~draw an equilibrium~~ polygon whose segments will be tangent to the curve of the corresponding linear arch, and indicate its position with sufficient exactness for practical purposes. See Fig. 370 The successive points of tangency A, m, n, etc. lie vertically under the points of division between the groups. This relation forms the basis of the graphical treatment of voussoir, or blockwork, arches.

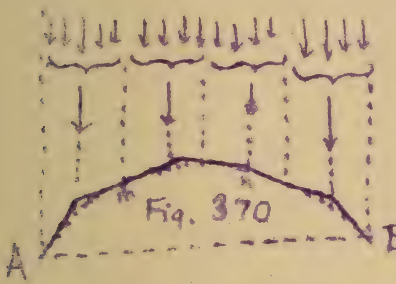


Fig. 370

341. TO PASS AN EQUIL. POLYGON THROUGH THREE ARBITRARY POINTS. Given a system of loads, it is required to draw an equilibrium for them through any three points, two of which may be considered as abutments, outside of the load-verticals, the third between the verticals of the first two. See Fig. 370. The loads P_1 etc. are given, with their verticals, while A, p, and B are the three points. Lay off the load line, and with any convenient pole, O_1, construct a force-diagram

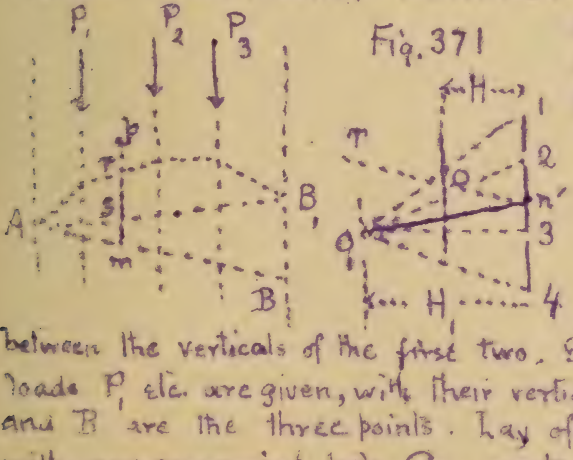


Fig. 371

and a corresponding preliminary equil. polygon, beginning at A. Its right abutment B_1, in the vertical through B, is thus found. $O_1 n'$ can now be drawn ∥ to AB_1. To determine n'. Draw $n'T$ ∥ to BA. The pole of the required equil. polygon must lie on $n'T$ (§ 338). Draw a vertical through p. The H of the required equil. polygon must satisfy the proportion $H : H_1 :: \overline{rs} : \overline{pm}$ (See § 338). Hence construct.

or compute H from this proportion and draw a vertical at a distance H from the load-line (on the left of the load-line here); its intersection with $n'T$ gives O the desired pole for which a force diagram may now be drawn. The corresponding equil. polygon beginning at the first point A will also pass through p and B; it is not drawn in the figure.

342. SYMMETRICAL CASE OF THE FOREGOING PROBLEM. If two points A and B are on a level, the third, p, on the middle vertical between them; and the loads (an even number) symmetrically disposed both in position and magnitude, about p, we may proceed ~~more simply~~, as follows: (Fig. 372)

Fig. 372

From symmetry n' must come at the middle of the load-line, of which we need lay off only the upper half. Take a convenient pole O_1 in the horizontal through n', and draw a half force diagram and a corresponding half equil. polygon (both dotted). The upper (horizontal) segment, bc, of the latter being produced, [illegible]

[illegible] determines the vertical CD containing the ce[illegible]

of gravity of the loads occurring on the left-hand half span (see §336). In the required equilibrium polygon the segment containing the point *p* must be horizontal, and its intersection ~~(both prolonged) with the first~~ segment must lie in CD. Hence determine this intersection, C, by drawing the vertical CD and a horizontal through *p*; then join CA, which is the first segment of the required equil. polygon. A ∥ to CA through 1 is the first ray of the corresponding force diagram, and determines the pole O on the horizontal through *n*′. Completing the force diagram for this pole (half of it only, here), the required equil. polygon is easily finished afterwards.

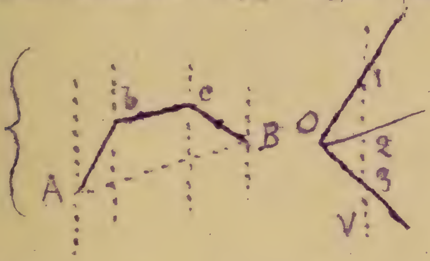
In treating symmetrical arches, symmetrically loaded, this construction will be of great use.

§343. TO FIND A SYSTEM OF LOADS UNDER WHICH A GIVEN EQUILIBRIUM POLYGON WOULD BE IN EQUILIBRIUM. FIG. 373. Let AB be the given equilibrium polygon. Through any point O as a pole draw a ∥ to each segment of the equil. polygon. Any vertical, as V, cutting these lines will have intercepted upon it a load-line 1 2 3, whose parts 1..2, 2..3 etc are proportional to the successive loads which, placed on the corresponding joints of the equil. polygon will be supported by it in equilibrium (unstable).

One load may be assumed and the others constructed.

A *hanging*, as well as a *standing*, equil. polygon may be dealt with in like manner, but will be in stable equilibrium.

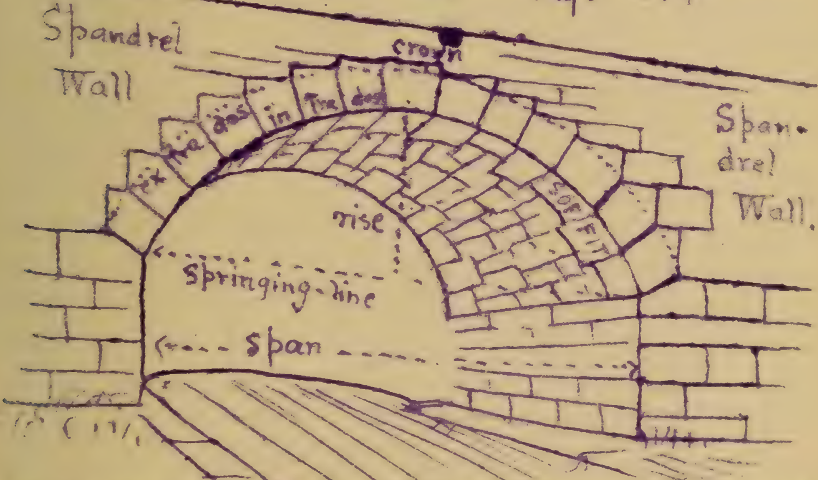
Fig. 373



Chap. X Right Arches of Masonry.

344. In an ordinary "right" stone-arch (i.e., one in which the faces are ⊥ to the axis of the cylindrical soffit, or under surface) the successive blocks forming the arch-ring are called voussoirs [voo-swars], the joints between them being planes which, prolonged, meet generally in one or more horizontal lines; e.g., those of a three-centred arch in three ∥ horizontal lines; those of a circular arch in one, the axis of the cylinder; etc. Elliptic arches are also used, sometimes. The inner concave surface is called the soffit, to which the radiating joints between the voussoirs are made perpendicular. The curved line in which the soffit is intersected

Fig. 374



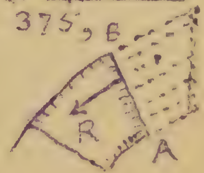
by a plane ⊥ to the axis of the arch is the INTRADOS. The curve in the same plane as the intrados, and bounding the outer extremities of the joints between the voussoirs is called the EXTRADOS.

Fig. 374 gives other terms in use in connection with a stone arch, and explains those already given.

345. MORTAR and FRICTION. As common mortar hardens very slowly, no reliance should be placed on its tenacity as an element of stability in arches of any considerable size; though hydraulic mortar, and thin joints of ordinary mortar can sometimes be depended on. Friction, however, between the surfaces of contiguous voussoirs, however, plays an essential part in the stability of an arch, and will therefore be considered.

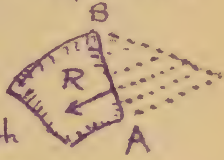
The stability of voussoir-arches must ∴ be made to depend on the resistance of the voussoirs to compression and to sliding upon each other; as also of the blocks composing the piers, the foundations of the latter being firm.

346. POINT OF APPLICATION OF THE RESULTANT PRESSURE between two consecutive voussoirs; (or pier blocks). Applying Navier's principle (as in flexure of beams) that the pressure per unit area on a joint varies uniformly from the extremity under greatest compression to the point of least compression (or of no compression); and remembering that negative pressures (i.e. tension) can not exist, as ~~they might~~ in a curved beam, we may represent the pressure per unit area at successive points of a joint (from the intrados toward the extrados, or vice versâ) by the ordinates of a straight line, forming the surface of a trapezoid or triangle, ~~in~~ which figure the foot of the ordinate of the centre of gravity is the point of application of the resultant pressure. Thus, Fig. 375,



where the least compression is supposed to exist at the intrados A, the pressures vary as the ordinates of a

Fig. 376



trapezoid, increasing to a maximum value at B, in the extrados. In Fig. 376, where the pressure is zero at B, and varies as the ordinates of a tri-angle, the resultant pressure acts through a point ONE THIRD the joint-length from A. Similarly in Fig. 377, it acts one third the joint length from B. Hence when the pressure is not zero at either edge the resultant pressure acts within the MIDDLE THIRD of the joint. Whereas, if the resultant pressure falls without the middle third, it shows that a portion Am of the joint, see Fig. 378, receives no pressure, i.e., the joint tends to open along Am.

Fig. 377

Fig. 378

Therefore that no joint tend to open, the resultant pressure must fall within the middle third.

It must be understood that the joint surfaces here dealt ~~with~~ are rectangles, seen edgewise in the figures.

347. FRICTION. By experiment it has been found the angle of friction (see § 156) for two contiguous voussoirs of stone or brick is about 30°; i.e., the coefficient of friction is $f = \tan. 30°$. Hence if the direction of the pressure exerted upon a voussoir by its neighbor makes an angle less than 30° with the NORMAL TO THE JOINT SURFACE, there is no danger of rupture of the arch by the sliding of one on the other; see Fig. 379

348. RESISTANCE TO CRUSHING. When the resultant pressure falls at its extreme allowable limit, viz. the edge of the middle third,

Fig. 379

the pressure per unit of area at n, Fig. 380, is <u>double</u> the mean pressure per unit of area. Hence, in designing an arch of masonry, we must be assured that at every joint (taking 10 as a factor of safety)

Fig. 380

$$\left\{\begin{matrix}\text{Double the mean pressure}\\ \text{per unit of area}\end{matrix}\right\} \text{must be less than } \frac{1}{10}C$$

C being the ultimate resistance to crushing, of the material employed (§ 201) (Modulus of Crushing.)

Since a lamina ONE FOOT thick will always be considered in what follows, careful attention must be paid to the units employed in applying the above test.

EXAMPLE. If a joint is 3 ft. by 1 foot, and the resultant pressure is 22.5 tons the mean pressure per sq. foot is

$$p = 22.5 \div 3 = 7.5 \text{ tons per sq. foot}$$

∴ its double = 15 tons per sq. foot = 208.3 lbs. per (sq. inch) which is much less than $\frac{1}{10}$ of C for most building stones; see § 203.

At joints where the resultant pressure falls at the middle, the max. pressure per square inch would be equal to the mean " " " "; but for safety it is best to assume that, at times, (from moving loads, or vibrations) it may move to the edge of the middle third, causing the max. pressure to be double the mean (per square inch).

349. THE THREE CONDITIONS OF SAFE EQUILIBRIUM for an arch of uncemented voussoirs

Recapitulating the results of the foregoing paragraphs, we may state, as follows, the three conditions which must be satisfied at every joint of arch-ring and pier, for any possible combination of loads upon the structure:

(1.) The resultant pressure must pass within the middle-third.

(2.) The resultant pressure must not make an angle > 30° with the normal to the joint.

(3.) The mean pressure per unit of area on the surface of the joint must not exceed $\frac{1}{20}$ of the Modulus of crushing of the material.

350. The TRUE LINEAR-ARCH, or SPECIAL EQUILIBRIUM POLYGON; and the resultant pressure at any joint. Let the weight of each voussoir and its load be represented by a vertical force passing through the centre of gravity of the two, as in Fig. 381.

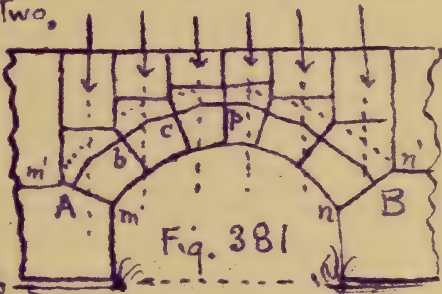


Fig. 381

Taking any two points A and B, A being in the first joint and B in the last; also a third point, p, in the crown joint (supposing such to be there, although generally a key-stone occupies the crown) through these three points can be drawn [§341] an equilibrium polygon for the loads given; suppose this equil. polygon nowhere passes outside of the arch-ring (the arch-ring is the portion between the intrados, m n, and the (dotted) extrados m'n') intersecting the joints at b, c, etc. Evidently if such be the case, and small metal rods (not round) were inserted at A, b, c, etc., so as to separate the arch-stones slightly, the arch would stand, though in unstable equilibrium, the piers being firm; and by a different choice of A, p, and B, it might be possible to draw other equilibrium polygons with segments cutting the joints within the arch-ring, and if the metal rods were shifted to these new intersections the arch would a-

gain stand (in unstable equilibrium).

In other words, if an arch stands, it may be possible to draw a great number of linear arches within the limits of the arch-ring, since three points determine an equilibrium polygon (or linear arch) for given loads. The question arises then: WHICH LINEAR ARCH IS THE LOCUS OF THE ACTUAL RESULTANT PRESSURES AT THE SUCCESSIVE JOINTS?

[Considering the arch-ring as an elastic curved beam inserted in firm piers (i.e. the blocks at the springing-line are incapable of turning) and having secured a close fit at all joints before the centering is lowered, the most satisfactory answer to this question is given in Prof. Greene's "Arches" p. 131; but the tedious computations there-employed may be most advantageously replaced by Prof. Eddy's graphic method ("New Constructions in Graphical Statics" published by Van Nostrand, New York 1877)

This method will be given in a subsequent chapter on Arch Ribs or Curved Beams; but for arches of masonry a much simpler procedure is sufficiently exact for practical purposes. This will now be presented]

When two elastic blocks touch at one edge, Fig. 382 their adjacent sides making a small angle with each other, and are then gradually pressed more and more forcibly together at the edge m, without altering their position angularly, the surface of contact becomes larger and larger, from the compression which ensues (see Fig. 383); while the resultant pressure between the blocks, first applied at the extreme edge m, has now advanced nearer the middle of the joint.

Fig. 382

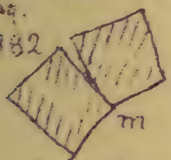
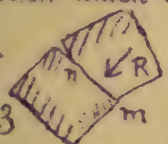


Fig. 383



With this in view we may reasonably deduce the following theory of the location of the TRUE

LINEAR ARCH (sometimes called the "line of pressures" and "curve of pressure") in an arch under given loading and with FIRM PIERS. (Whether the piers are really unyielding, under the oblique thrusts at the springing-line, is a matter for subsequent investigation.

351. LOCATION OF THE TRUE LINEAR ARCH. Granted that the voussoirs have been closely fitted to each other over the centering (sheets of lead are sometimes used in the joints to make a better distribution of pressure); and that the piers are firm; and that the arch can stand at all without the centering; then in the mutual accommodation between the voussoirs, as the centering is lowered, the resultant of the pressures distributed over any joint, if at first near the extreme edge of the joint, advances nearer to the middle as the arch settles to its final position of equilibrium under its load; and hence the

352 PRACTICAL CONCLUSIONS

I. If for a given arch and loading, with firm piers, an equilibrium polygon can be drawn (by proper selection of the points A p and B Fig 381) entirely within the <u>middle third</u> of the arch ring, not only will the arch stand, but the resultant pressure at every joint will be within the middle third (Condition I, § 349); and among all possible equilibrium polygons which can be drawn within the middle third, that is the "true" one which most nearly co-incides with the middle line of the arch-ring

II. If (with firm piers, as before) no equilibrium polygon can be drawn within the middle third, and only one within the arch-ring at all, the arch may stand but chipping and spawling are likely to occur at the edges of the joints. The design should ∴ be altered.

III. If no equilibrium polygon can be drawn within the arch-ring, the design of either the arch or the load-

ing must be changed; since, although the arch may stand, from the resistance of the spandrel walls, such a stability must be looked upon as precarious and not countenanced in any large important structure. (Very frequently, in small arches of brick and stone, as they occur in buildings, the cement is so tenacious that the whole structure is virtually a single continuous mass)

When the "true" linear arch has once been determined, the amount of the resultant pressure on any joint is given by the length of the proper RAY in the force diagram.

ARRANGEMENT OF DATA FOR GRAPHIC TREATMENT.

353. CHARACTER OF LOAD. In most large large stone arch bridges the load (permanent load) does not consist exclusively of masonry up to the road-way but partially of earth filling above the masonry, except at the faces of the arch where the spandrel walls serve as retaining walls to hold the earth; Fig 384. If the intrados is a half circle or half-ellipse, a compactly-built masonry backing is carried up beyond the springing-line to AB about 60° or 45° from the crown; Fig. 385; so that the portion of arch ring below AB may be considered as part of the abutment, and thus AB is the virtual springing-line, for graphic treatment.

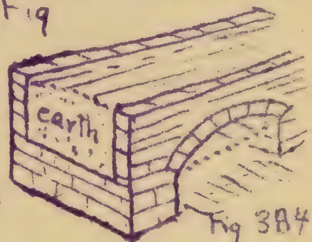


Fig 384

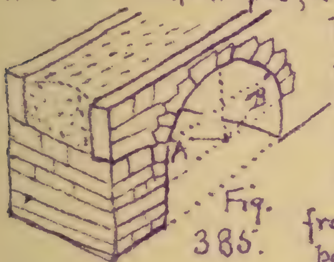


Fig. 385.

Sometimes, to save filling, small arches are built over the haunches of the main arch, with earth placed over them as shown in Fig. 386.

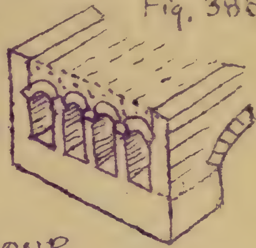


Fig. 386

In any of the preceding cases it is customary to consider that, on account of the bonding of the stones in the arch shell, the loading at a given distance from the crown ~~to be~~ is uniformly distributed over the width of the roadway.

§ 354 REDUCED LOAD-CONTOUR.

In the graphical discussion of a proposed arch we consider a lamina one FOOT thick, this lamina being vertical and ⊥ to the axis of the arch; i.e. the lamina is ∥ to the spandrel walls. For graphical treatment, equal areas of the elevation (see Fig. 387) of this lamina must represent equal weights. Taking the material of the arch-ring as a standard, we must find for each point p of the extrados an imaginary height z of the arch-ring material, which would give the same pressure (per running horizontal foot) at that point as ~~[illegible]~~ to the actual load above that point. A number of such ordinates, each measured vertically upward from the extrados determine points in the "REDUCED LOAD-CONTOUR", i.e. the imaginary line, AM, the area between which and the extrados of the arch-ring represents a homogeneous load of the same density as the arch-ring, and equivalent to the actual load (above extrados), vertical by vertical

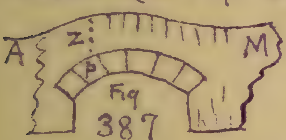


Fig 387

355. EXAMPLE OF REDUCED LOAD-CONTOUR

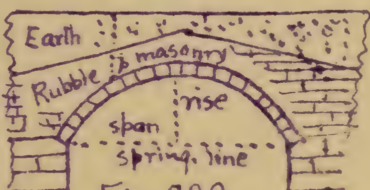


Fig. 388

Fig. 388. Given an arch-ring of granite (heaviness = 170 lbs. per cubic foot) with a dead load of rubble (heav. = 140.) and earth (heav. = 100), distributed as in figure. At the point p, of the extrados, the depth 5 feet of rubble is equivalent to a depth

of $\left[\frac{140}{170} \times 6\right] = 4.1$ ft. of granite ; while the 6 feet of earth is equivalent to $\left[\frac{100}{170} \times 6\right] = 3.5$ feet of granite. Hence the REDUCED LOAD-CONTOUR has an ordinate, above p, of 7.6 feet. That is, for each of several points of the arch-ring extrados reduce the rubble ordinate in the ratio of 170 : 140, and the earth ordinate in the ratio 170 : 100 and add the results, setting off the sum vertically from the point in the extrados. In this way Fig. 389 is obtained, and the area there given is to be treated as representing homogeneous granite one foot thick. This of course now includes the arch-ring also.

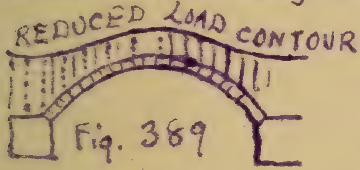


Fig. 389

356. LIVE LOADS. In discussing a rail-road arch bridge the "live load" (a train of locomotives, e.g., to take an extreme case) can not be disregarded, and for each of its positions we have a separate REDUCED LOAD-CONTOUR.

Example. Suppose the arch of Fig. 388 to be 12 ft. wide (not including spandrel walls) and that a train of locomotives weighing 3000 lbs. per running foot of track covers one half of the span. Uniformly distributed laterally over the width, 12 ft, this rate of loading is equivalent to a masonry load of one foot high and 250 lbs. per cub. ft. heaviness; i.e., is equivalent to a height of 1.4 ft. of granite masonry $\left[\text{since } \frac{250}{170} \times 1.0 = 1.4\right]$ over the half span considered. Hence from Fig. 390 we obtain Fig. 391 in an obvious manner. Fig. 391 is now ready for graphic treatment.

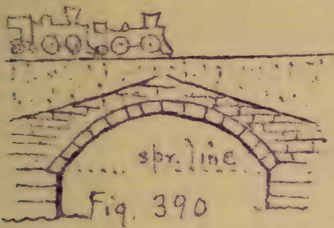


Fig. 390

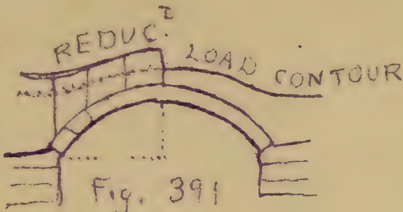


Fig. 391

357. PIERS AND ABUTMENTS. In a series of equal arches the pier between two consecutive arches bears simply the weight of the two adjacent SEMI-arches, plus the load immediately above the pier, and ∴ does not need to be as large as the abutments of the first and last arches, since these latter must be prepared to resist the oblique thrusts of their arches without help from the thrust of another arch on the other side.

In a very long series of arches it is sometimes customary to make a few of the intermediate piers large enough to act as abutments. These are called "abutment piers", and in case one arch should fall, no others would be lost except those occurring between the same two abutment piers as the first. See Fig. 392. A, A', etc are abut. piers.

Fig. 392.

GRAPHICAL TREATMENT OF ARCH.

358. Having found the "reduced load-contour," as in preceding paragraphs, for a given arch and load, we are ready to proceed with the graphic treatment; i.e., the first given, or assumed, form and thickness of arch-ring is to be investigated with regard to stability. It may be necessary to treat, separately, a lamina under the spandrel wall, and one under the interior loading.

CASE I. SYMMETRICAL ARCH AND SYMMETRICAL LOADING. (The "steady" (permanent) or "dead" load on an arch is usually symmetrical) Fig. 393.

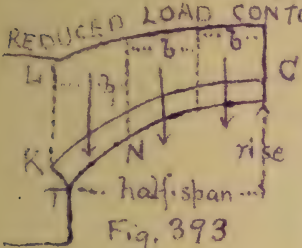


Fig. 393

From symmetry we need deal with only one half (say the left) of the arch and load. Divide this semi-arch and load into six or ten divisions by vertical lines; these divisions are considered as trapezoids and should have the same horizontal width = b (a convenient whole number of feet) except the last one, LKN, next the abutment, and this is a pentagon of a different width b_1 (the remnant of the horizontal distance LC) The weight of masonry in each division is equal to

(the area of division) X (unity thickness of lamina) X (weight of a cubic unit of arch-ring)

and is applied through the centre of gravity of the division. The area of a trapezoid, Fig. 394, is $\frac{1}{2} b (h_1 + h_2)$,

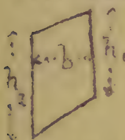


Fig. 394

and its centre of gravity may be found, Fig. 395, by the construction of Prob. 6. in § 26; or by § 27a.

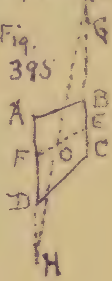


Fig. 395

The weight of the pentagon LN Fig. 393, and its line of application (through centre of gravity) may be found by combining results for the two trapezoids into which it is divided by a vertical through T.

Now consider A, the middle point of the abutment joint, as the starting point of an equilibrium polygon (or abutment of a linear arch) for the given loading, and require that this equilibrium polygon shall pass through

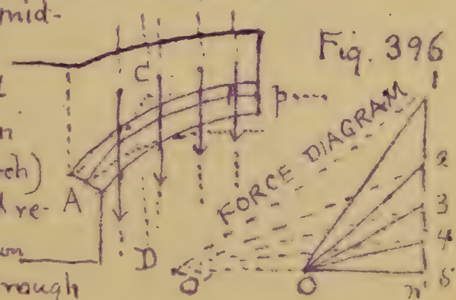


Fig. 396

p, the middle of the crown joint, and through the middle of the abutment joint on the right (not shown in figure).

Proceed as in § 342, thus determining the polygon A p for the half-arch. Draw joints in the arch ring through those points where the extrados is intersected by the verticals separating the divisions (not the gravity verticals). The points in which these joints are cut by the segments of the equilibrium polygon, Fig. 397, are the points of application in these joints respectively of the resultant pressures on them, (IF this is the true linear arch for this arch and load) while the amount and direction of each such pressure is given by the proper ray in the force-diagram.

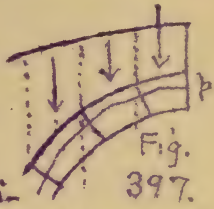


Fig. 397.

If at any joint so drawn the linear arch (or equil. pol.) passes outside the middle third of the arch-ring, the point A, or p, (or both) should be judiciously moved (within the middle third) to find if possible a linear arch which keeps within limits at all joints. If this is found impossible, the thickness of the arch-ring may be increased at the abutment (giving a smaller increase toward the crown) and the desired result obtained; or a change in the distribution or amount of the loading, if allowable may gain this object. If but one linear arch can be drawn within the middle third, it may be considered the "true" one; if several, the one most nearly co-inciding with the middles of the joints (see §§ 351 and 352) is so considered

§ 359. Case II

UNSYMMETRICAL LOADING on a SYMMETRICAL ARCH; (e.g. arch with live load covering one half-span as in Figs. 390 and 391) Here we must evidently use a full force diagram, and the full elevation of the arch-ring and load. See Fig. 398. Select three

points A, p, and B, as follows, to determine a trial equilibrium polygon: Select A at the lower limit of the middle third of the abutment-joint, at that end of the span which is the more

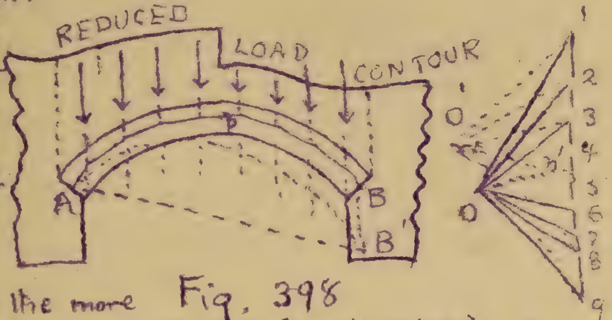


Fig. 398

heavily loaded; in the other abutment joint take B at the upper limit of the middle third; and take p in the middle of the crown joint. Then by § 341 draw an equilibrium polygon (i.e., a linear arch) through these three points for the given set of loads, and if it does not remain within the middle third, try other positions for A, p and B, within the middle third. As to the "true linear arch" alterations of the design, etc., the same remarks apply as already given in Case I. Very frequently it is not necessary to draw more than one linear-arch, for a given loading, for even if one could be drawn nearer the middle of the arch-ring than the first, that fact is most always apparent on mere inspection, and the one already drawn (if within middle third) will furnish values sufficiently accurate for the pressures on the respective joints, and their direction angles.

360.

The design for the arch-ring and loading is not to be considered satisfactory until it is ascertained that for the dead load and any possible combination of live-load (in addition) the pressure at any joint is

(1.) Within the middle third of that joint;

(2.) At an angle of < 30° with the normal to joint-surface.

(3.) Of a mean pressure per square inch not > than $\frac{1}{20}$ of the ultimate crushing resistance. (See § 348)

§ 360] ABUTMENTS. The abutment should be compactly and solidly built, and is then treated as a single rigid mass. The pressure of the lowest voussoir upon it (considering a lamina one foot thick) is given by the proper ray of the force diagram (0..1 e.g. in Fig. 396) in amount and direction. The stability of the abutment will depend on the amount and direction of the resultant obtained by combining that pressure P_a with the weight G of the abutment and its load; see Fig. 399. Assume a probable width RS for the abutment and compute the weight G of the corresponding abutment OBRS and load MNBO, and find the centre of gravity of the whole mass C. Apply G in the vertical through C, and combine it with P_a at their intersection D. The resultant P should not cut the base RS in a point beyond the middle third (or, if this rule gives too massive a pier, take such a width that the pressure per square inch at S shall not exceed a safe value as computed from § 362) After one or two trials a satisfactory width can be obtained. We should also be assured that the angle PDG is less than 30°. The horizontal joints above RS should also be tested as if they were, in turn, the lowest base, and if necessary may be inclined (like mn) to prevent slipping. On no joint should the maximum pressure per square inch be > than $\frac{1}{10}$ the crushing strength of the cement. Abutments of firm natural rock are of course to be preferred where they can be had. If water penetrates under an abutment its buoyant effort lessens the weight of the latter to a considerable extent.

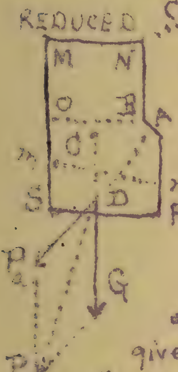


Fig. 399.

362. MAXIMUM PRESSURE PER UNIT OF AREA WHEN THE RESULTANT PRESSURE FALLS AT ANY GIVEN DISTANCE FROM THE MIDDLE; according to Navier's theory of the distribution of the pressure; see § 346.

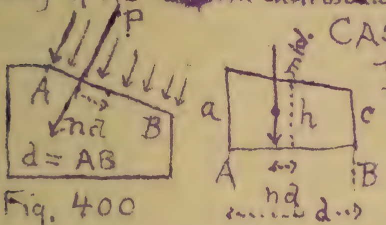


Fig. 400

CASE I. Let the resultant pressure P fall within the middle third, a distance $= nd$ $(< \frac{1}{6}d)$ from the middle of joint. Then we have the following relations: p (the mean press. per sq. in), p_m (max. press. per sq. in), and p_n (least " " " ") are proportional to the lines h (mid. vert.), a (max. base), and c (min. base) respectively, of a trapezoid (Fig. 400) through whose centre of gravity P acts. But (§ 26)

$$nd = \frac{h}{6} \cdot \frac{a-c}{a+c} \text{ i.e. } n = \frac{1}{6}\,\frac{a-h}{h} \text{ or } a = h(6n+1)$$

$$\therefore p_m = p(6n+1)$$ Hence the following table:

If $nd =$	$\frac{1}{6}d$	$\frac{1}{9}d$	$\frac{1}{18}d$	then the max.
press. $p_m =$	2	$5/3$	$4/3$	times the mean

Case II. Let P fall outside the mid. third, a distance $= nd$ $(> \frac{1}{6}d)$ from middle of joint. Here since the joint is not considered capable of withstanding tension we have a triangle, instead of a trapezoid, Fig. 401. First compute the mean press. per sq. in

$$p = \frac{P \text{ (lbs.)}}{(1-2n)\,18\,d^{\text{inches}}}$$ or from this table: (lamina one foot thick)

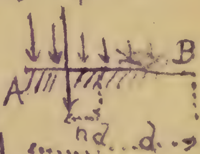


Fig. 401

For $nd =$	$\frac{4}{18}d$	$\frac{5}{18}d$	$\frac{6}{18}d$	$\frac{7}{18}d$	$\frac{8}{18}d$	$\frac{9}{18}d$
$p =$	$\frac{1}{10}\frac{P}{d}$	$\frac{1}{8}\frac{P}{d}$	$\frac{1}{6}\frac{P}{d}$	$\frac{1}{4}\frac{P}{d}$	$\frac{1}{2}\frac{P}{d}$	infinity

(d in inches.)

and then max. press. $= p_m = 2p$.

Chap. XI. ARCH-RIBS.

364. DEFINITIONS and ASSUMPTIONS. An arch-rib (or elastic-arch, as distinguished from a block-work arch) is a rigid curved beam, either solid, or built up of pieces like a truss (and then called a braced arch) the stresses in which, under a given loading and with prescribed mode of support it is here proposed to determine. The rib is supposed symmetrical about a vertical plane containing its axis or middle line, and the Moment of Inertia of any cross section is understood to be referred to a gravity axis of the section, which (the axis) is perpendicular to the said vertical plane. It is assumed that in its strained condition under a load, its shape differs so little from its form when unstrained that the change in the abscissa or ordinate of any point in the rib axis (a curve) may be neglected when added (algebraically) to the co-ordinate itself; also that the dimensions of a cross-section are small compared with the radius of curvature at any part of the curved axis, and with the span.

365. MODE OF SUPPORT. Either extremity of the rib may be hinged to its pier (which gives freedom to the end-tangent-line to turn in the vertical plane of the rib when a load is applied); or may be fixed, i.e. so built-in, or bolted rigidly to the pier, that the end-tangent-line is incapable of changing its direction when a load is applied. A hinge may be inserted anywhere along the rib, and of course destroys its rigidity, or resistance to bending at that point. A hinge having its pin horizontal and ⊥ to the axis of the rib is meant. Evidently no more than three such hinges could be introduced along an arch-rib between two piers; un-

is designed to be a hanging structure, acting as a suspension cable.

366. ARCH RIB AS A FREE BODY. In considering the whole rib free it is convenient, for graphical treatment, that no section be conceived made at its extremities, if fixed; hence in dealing with that mode of support the end of the rib will be considered as having a rigid prolongation reaching to a point vertically above or below the pier-junction, an unknown distance from it, and there acted on by a force of such unknown amount and direction as to preserve the actual extremity of the rib and its tangent line in the same position and direction as they really are. As an illustration of this Fig. 402 shows FREE an arch-rib ONB, with its extremities O and B fixed in the piers, with no hinges, and bearing two loads P_1 and P_2. The other forces of the system holding it in equilibrium

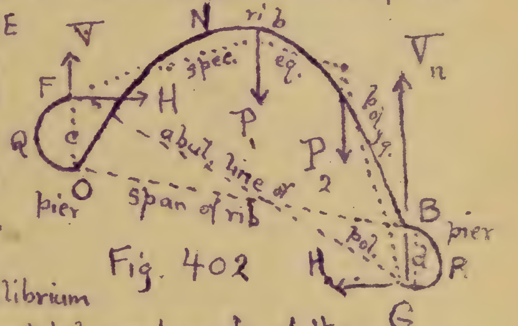


Fig. 402

are the horizontal and vertical components of the pier reactions (H V H_n and V_n), and in this case of fixed ends each of these two reactions is a single force not intersecting the end of the rib, but cutting the vertical through the end in some point F (on the left; and in G on the right) at some vertical distance c (or d) from the end. Hence the utility of these imaginary prolongations OQF, and BRG, the pier being supposed removed. Compare Figs. 348 and 350.

The imaginary points, or hinges, F and G, will be called abutments being such for the special equilibrium polygon (dotted broken line), while O and B

are the real ends of the curved beam, or rib.

In this system of forces there are five unknowns viz.: V, V_n, $H = H_n$, and the distances c and d. Their determination by analysis, if the rib is a circular arc is extremely intricate and tedious; but by graphical statics (Prof. Eddy's method; see p. 199 for reference) it is comparatively simple and direct and applies to *any shape* of rib, being sufficiently accurate for practical purposes. This method consists of constructions leading to the location of the "special equilibrium polygon" and its force diagram. In case the rib is hinged to the piers, the reactions of the latter act through these hinges, Fig. 403, i.e., the abutments of the special equilibrium polygon coincide with the ends of the rib O and B, and for a given rib and load the unknown quantities are only three V, V_n, and H; (strictly there are four; but $\Sigma X = 0$ gives $H_n = H$) The solution by analytics is possible only for ribs of simple algebraic curves and is long and cumbrous; whereas Prof. Eddy's graphic method is comparatively brief and simple and is applicable to any shape of rib whatever.

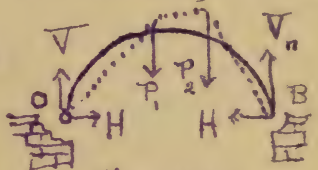


Fig. 403

367. UTILITY OF THE SPECIAL EQUILIBRIUM POLYGON and its force diagram. The use of locating these will now be illustrated [See § 332] As proved in §§ 332 and 334 the compression in each "rod" or segment of the "special equilibrium polygon is the anti-stress resultant of the cross sections in the corresponding portion of the beam, rib, or other structure, the value of this compression (in lbs. or tons) being measured by the length of the parallel ray in the force diagram. Suppose that in some way (to be explained sub-

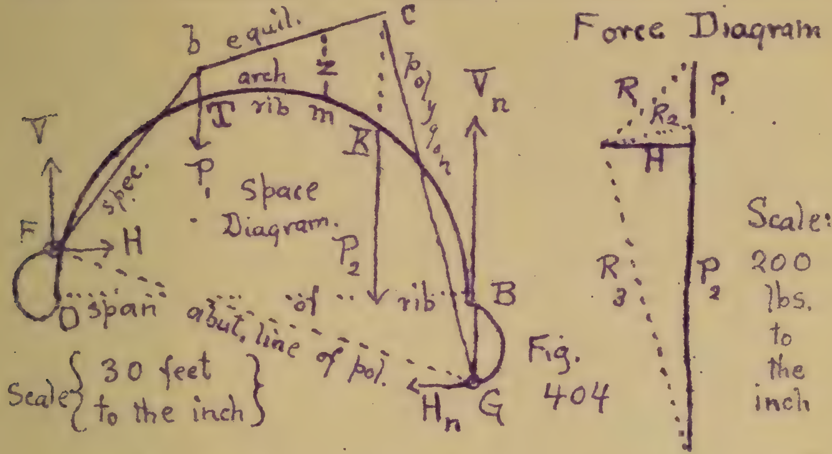


Fig. 404

sequently) the special equilibrium polygon and its force diagram for the arch-rib in Fig. 404 having FIXED ENDS O and B and no hinges; required the elastic stresses in any cross-section of the rib as at m. This cross section m lies in a portion IK, of the rib, corresponding to the rod or segment bc of the equilibrium polygon, hence its anti-stress-resultant is a force R_2 acting in the line bc, and of an amount given in the force-diagram. Now R_2 is the resultant of V, H, and P_1, which with the elastic forces at m form a system in equilibrium, shown in Fig. 405; the portion FOTm being considered free. Hence taking the tangent line and the normal at m as axes we should have Σ(tang. comps.) $= 0$; Σ(norm. comps.) $= 0$; and Σ(moms. about gravity axis of the section at m) $= 0$ and could thus find the unknowns

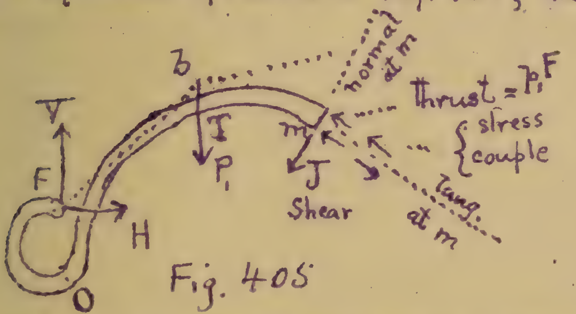


Fig. 405

p_1, p_2, and J, which appear in the expressions p_1F the thrust, $\frac{p_2 I}{e}$ the moment of the stress-couple, and J the ~~shear~~ shear. These elastic stresses are classified as in § 295, which see. p_1 and p_2 are lbs. per square inch, J is lbs., e is the distance from the horizontal gravity axis of the section to the outermost element of area, (where the compression or tension is p_2 lbs. per sq. inch) while I is the "moment of inertia" of the section about that gravity axis. [See §§ 247 and 295; also § 85] Graphics, however, gives us a more direct method, as follows: Since R_2, in the line bc, is the equivalent of V, H, and P_1, the stresses at m will be just the same as if R_2 acted directly upon a lateral prolongation of the rib at T (to intersect bc Fig. 405) as shown in Fig. 406, this prolongation Tb taking the place of TOF in Fig. 405. The force diagram is also reproduced here. Let a denote the ⊥ from m's gravity axis upon bc, and z the vertical intercept between m and bc.

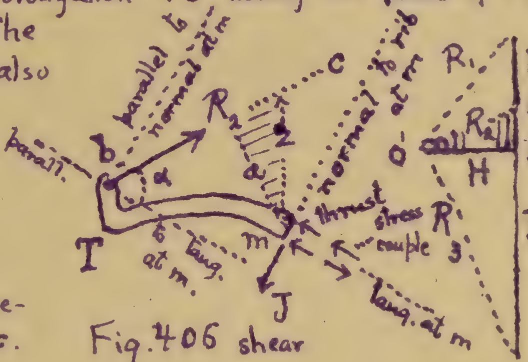


Fig. 406

For this imaginary free body, we have from Σ(tang. compons.) = 0, $R_2 \cos\alpha = p_1F$
and from Σ(norm ") = 0 $R_2 \sin\alpha = J$
while from Σ(moms. about the gravity axis of m) = 0, we have $R_2 a = \frac{p_2 I}{e}$

But from the two similar triangles (shaded; one of them is in force diagram) $a : z :: H : R_2$ $\therefore R_2 a = Hz$,

whence we may rewrite these relations as follows (with a general statement) viz.:

IF THE SPECIAL EQUILIBRIUM POLYGON AND ITS FORCE DIAGRAM HAVE BEEN DRAWN for a given arch-rib, of given mode of support, and under a given loading, then in any cross section of the rib, we have (F = area of section)

(1.) The THRUST, i.e. p_1F = { The projection of the proper ray (of the ~~force~~ diagram) upon the tang. line of the rib drawn at the given section }

(2.) The SHEAR, i.e. J (upon which depends the shearing stress in the web) (See §§ 253 and 256) = { The projection of the proper ray (of the force diagram) upon the normal to the rib curve at the given section }

(3.) The MOMENT of the stress couple, i.e. $\frac{p_2 I}{e}$ = { The product (H z) of the H (or pole-distance) of the force-diagram by the vertical of the gravity axis of the section from the spec. equilibrium polygon }

By the "proper ray" is meant that ray which is parallel to the segment (of the equil. polygon) immediately under or above which the given section is situated. Thus in Fig. 404, the proper ray for any section on TK is R_2; on KB, R_3; on TO, R_1. The projection of a ray upon any given tangent or normal, is easily found by drawing through end of the ray a line ⊥ to the tangent (or normal); the length between these ⊥'s on the tangent (or normal) is the force required (by the scale of the force diagram). We may thus construct a shear diagram, and a thrust diagram for ~~a given case~~, while the successive

vertical intercepts between the rib and special equilibrium polygon form a moment-diagram.

368. It is thus seen how a location of the special equilibrium polygon, and the lines of the corresponding force diagram, lead directly to a knowledge of the stresses in all the cross-sections of the curved beam under consideration, bearing a given load; or, vice versâ, leads to a statement of conditions to be satisfied by the dimensions of the rib, for proper security.

It is here supposed that the rib has sufficient lateral bracing (with others which lie parallel with it) to prevent buckling sideways in any part like a long column. Before to the complete graphical analysis of the different cases of arch-ribs, it will be necessary to devote the next few paragraphs to developing a few analytic relations in the theory of flexure of a curved beam, and to giving some processes in "graphical arithmetic."

369. CHANGE IN THE ANGLE BETWEEN TWO CONSECUTIVE RIB TANGENTS when the rib is loaded, as compared with its value before loading. Consider any small portion (of an arch rib) included between two consecutive cross-sections; Fig. 407. Let $EA_2 = ds$, be the original length of this portion of the rib axis. The length of all the fibres (∥ to rib-axis) was originally $= ds$ (nearly) and the two consecutive tangent-lines, at E and A, made an angle $= d\theta$ originally. While under strain, however, all the fibres are shortened <u>equally</u> an amount dl' by the uniformly distributed tangential thrust, but are <u>unequally</u> shortened (or lengthened; according as they are on one side or the other of the gravity axis E, or A, of the section) by the system of forces making what we call the "stress couple", among which the stress at the distance e from the gravity-axis A of the section is called p_2

per square inch; so that the tangent line at B now takes the direction BD instead of BC (we suppose the section at E to remain fixed, for convenience, since the change of angle between the two tangents depends on the stresses acting, and not on the changed position in space, of this part of the rib) and hence

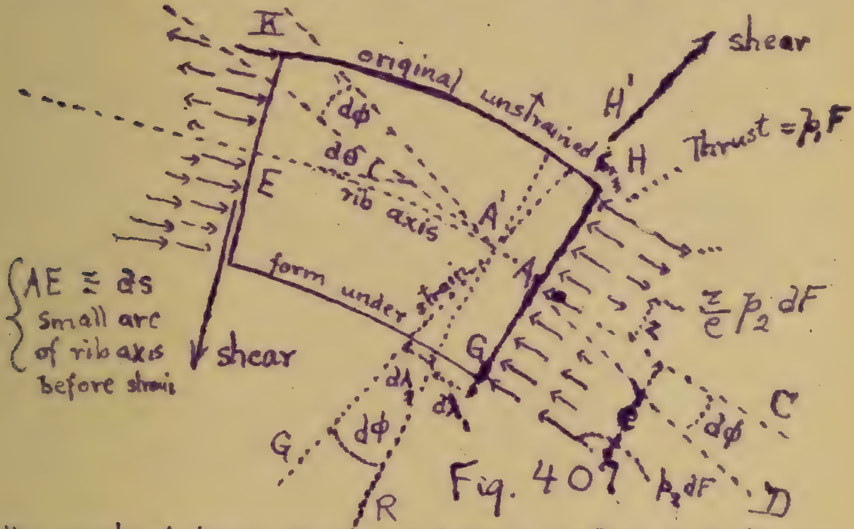


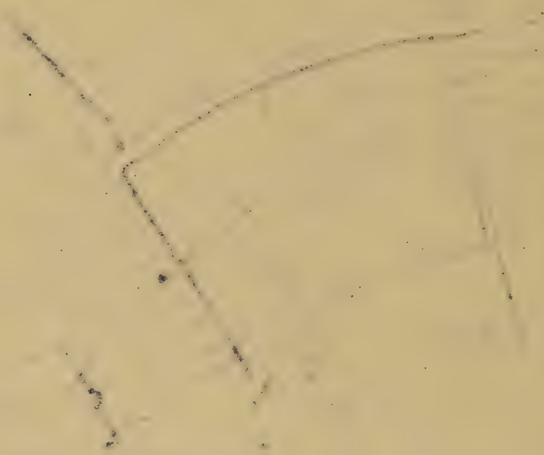
Fig. 407

the angle between the tangent-lines at E and A (originally $= d\theta$) is now increased by an amount $CAD = d\phi$ (or $G'A'R = d\phi$); GH' is the new position of GH. We obtain the value of $d\phi$ as follows: The shortening of the fibre at G at distance e from A, due to the force $p_2 dF$, is $d\lambda_2$, and from § 201 eq. (1) we have $d\lambda_2 = \frac{p_2 ds}{E}$

But, geometrically, $d\lambda_2$ also $= e\, d\phi$ ∴

$$E e\, d\phi = p_2 ds \;\ldots\ldots\; (1.)$$

But, letting M denote the moment of the stress-couple at section A (M depends on the loading, mode of support, etc. in any particular case) we know from § 295 eq. (6) that $M = \frac{p_2 I}{e}$ and hence by sub-



stitution in (1) we have $d\phi = \frac{M\,ds}{EI}$ ...(2.)

370. TOTAL CHANGE [i.e. $\int d\phi$] IN THE ANGLE BETWEEN THE END-TANGENTS OF A RIB, before and after loading.

Take the example in Fig. 408 of a rib fixed at one end and hinged at the other. When the rib is unstrained (as it is supposed to be, on the left, its own weight being neglected; it is not supposed sprung into place, but is entirely without strain)

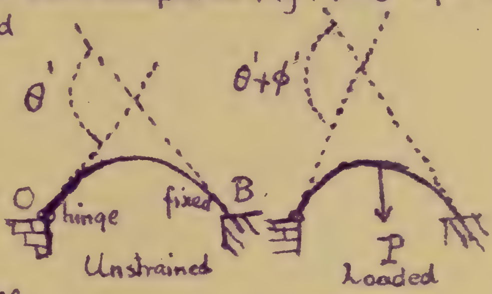


Fig. 408

then the angle between the end-tangents has some value $\theta' = \int_0^B d\theta$ = the sum of the successive small angles $d\theta$ for each element ds of the rib curve (or axis). After loading, [on the right] Fig. 408, this angle has increased having now a value

$$\theta' + \int_0^B d\phi \text{, i.e., a value } = \theta' + \int_0^B \frac{M\,ds}{EI} \quad \ldots\text{(I.)}$$

§ 371 EXAMPLE OF EQUATION (I.) in ANALYSIS

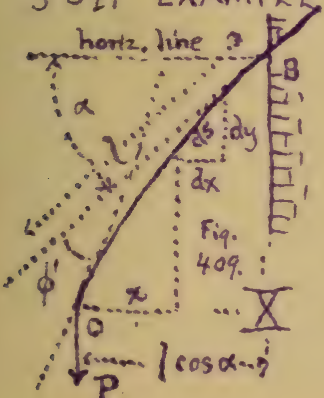


Fig. 409.

A straight, homogeneous, prismatic beam, Fig. 409, its own weight neglected, is fixed obliquely in a wall. After placing a load P on the free end, required the angle between the end-tangents. This was zero before loading ∴ its value after loading is

$$= 0 + \phi' = 0 + \frac{1}{EI}\int_0^B M\,ds$$

By considering free ~~a~~ portion between O and any ds of the beam, we find that $M = Px$ = mom. of the stress couple. The flexure is so slight that the angle between any ds and its dx is still practically $= \alpha$ and $\therefore\ ds = dx \sec\alpha$. Hence by substitution in eq (I) we have

$$\phi' = \frac{1}{EI}\int_0^B M\,ds = P\sec\alpha\int_0^{l\cos\alpha} x\,dx = \frac{P\sec\alpha}{EI}\left[\frac{x^2}{2}\right]_0^{l\cos\alpha}$$

$$\therefore\ \phi' = \frac{P\cos l^2}{2EI} \quad \text{[Compare with §237]}$$

It is now apparent that if both ends of an arch rib are fixed, when unstrained, and the rib be then loaded (within elastic limit and deformation slight) we must have $\int_0^B (M\,ds \div EI) = \text{zero}$, since $\phi' = 0$

372. PROJECTIONS OF THE DISPLACEMENT OF ANY POINT OF A LOADED RIB RELATIVELY TO ANOTHER POINT AND THE TANGENT LINE AT THE LATTER. Let O be the point whose displacement is considered and B the other point. Fig. 410

If B's tangent-line is fixed while the extremity O is not supported in any way (Fig. 410) then a load P is put on, O is displaced to a new position O_n. With O as an origin and

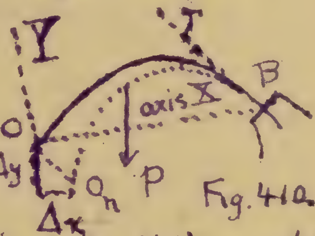


Fig. 410

OB as the axis of X, the projection of the displacement OO_n upon X will be called Δx, that upon Y, Δy.

In the case in Fig. 410, O's displacement with respect to B and its tangent-line BT, is also its absolute displacement in space, since neither B nor BT has moved as the rib changes form under the load. In Fig. 411 however, where the extremities O and B are both

hinged to piers, or supports, the dotted line showing its form when deformed under a load. The hinges are supposed immovable, the rib being free to turn about them without friction. The dotted line is the changed form under a load, and the absolute displacement of O is zero; but not so its displacement relatively to B and B's tangent BT, for BT has moved to a new position (dotted). To find this relative displacement, the new concave curve of the rib superposed on the old in such a way that B and BT may coincide with their original positions, Fig. 412. It is now seen that O's displacement relatively to B and BT is not zero but $= OO_n$, but has a small Δx but a comparatively large Δy (In fact for this case of hinged ends, piers immovable, rib continuous between them, and deformation slight, we shall write $\Delta x =$ zero as compared with Δy, the axis X passing through OB)

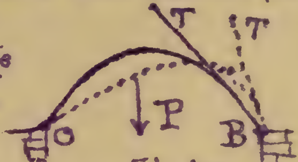


Fig. 411

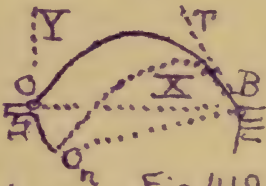


Fig. 412

373. VALUES OF THE X AND Y PROJECTIONS OF O's DISPLACEMENT RELATIVELY TO B AND B's TANGENT;

the origin being taken at O. Fig. 413. Let the co-ordinates of the different

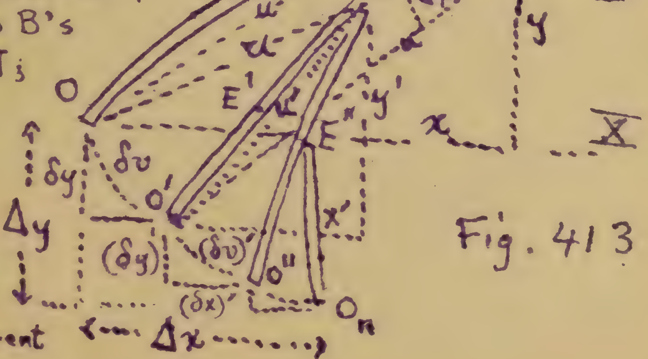


Fig. 413

points E, D, C, etc. of the rib, referred to O and an arbitrary X axis, be x and y, their radial distances from O being u (i.e. u for C, u′ for D, etc.; in general, u) O E D C is the unstrained form of the rib, while O_n E" D' C B is its form under some loading, i.e. under strain (The superposition above mentioned (§ 372) is supposed already made) if necessary, so that BT is tangent at B to both forms). Now conceive the rib OB to pass into its strained condition by the successive bending of each ds in turn The straining or bending of the first ds, BC, through the small angle $d\phi$ (dependent on the moment of the stress couple at C in the strained condition) causes the whole finite piece OC to turn about C as a centre through the same small angle $d\phi$; hence the point O describes a small linear arc OO′ = δv, whose radius = u the hypothenuse of the x and y of C, and whose value ∴ is $\delta v = u\,d\phi$

Next let the section D, now at D′, turn through its proper angle $d\phi'$ (dependent on its stress-couple) carrying with it the portion D′O′, into the position D′O″, making O′ describe a linear arc O′O″ = $(\delta v)' = u' d\phi'$, in which u' = the hypothenuse on the x′ and y′ (of D); (the deformation is so slight that the co-ordinates of the different points referred to O and X are not appreciably affected). Thus each section having been allowed to turn through the angle proper to it, O finally reaches its position, O_n, of displacement. Each successive δv, or linear arc described by O, has a shorter radius. Let δx, $(\delta x)'$, etc. represent the projections of the successive (δv)'s upon the axis X; and similarly δy, $(\delta y)'$ etc. upon the axis Y. Then the total X projection of the curved line $O \ldots O_n$ will be

$$\Delta x = \int \delta x \quad \text{and similarly} \quad \Delta y = \int \delta y \quad \ldots (1.)$$

But $\delta v = u\,d\phi$, and from similar right-triangles $\delta x : \delta v :: y : u$ and $\delta y : \delta v :: x : u$

$\therefore \delta x = y\,d\phi$ and $\delta y = x\,d\phi$; whence [see (1) and (2) of § 369]

$$\Delta x = \int \delta x = \int y\,d\phi = \int_0^B \frac{M y\,ds}{EI} \quad \ldots (II.)$$

and

$$\Delta y = \int \delta y = \int x\,d\phi = \int_0^B \frac{M x\,ds}{EI} \quad \ldots (III.)$$

If the rib is homogeneous E is constant, and if it is of constant cross-section, all sections being similarly cut by the vertical plane of the rib's axis (i.e., if it is a "curved prism") I, the moment of inertia is also constant.

374. RECAPITULATION OF ANALYTICAL RELATIONS, for reference.

TOTAL CHANGE IN ANGLE between tangent-lines at O and B $\Big\} = \int_0^B \frac{M\,ds}{EI} \quad \ldots (I.)$

THE X-PROJECTION of O's DISPLACEMENT RELATIVELY TO B AND B's ~~tangent-line~~; (the origin being at O and the axes X and Y ⊥ to each other) $\Big\} = \int_0^B \frac{M y\,ds}{EI} \quad \ldots (II.)$

THE Y-PROJECTION of O's DISPLACEMENT, etc. as above $\Big\} = \int_0^B \frac{M x\,ds}{EI} \quad \ldots III.$

Here x and y are the co-ordinates of points in the rib-curve, ds an element of that curve, M the moment of the stress-couple in the corresponding section as induced by the loading, or constraint, of the rib.

374 a. RESUMÉ OF THE PROPERTIES OF EQUILIBRIUM POLYGONS AND THEIR FORCE DIAGRAMS FOR SYSTEMS OF VERTICAL LOADS. See §§ 335 to 343.

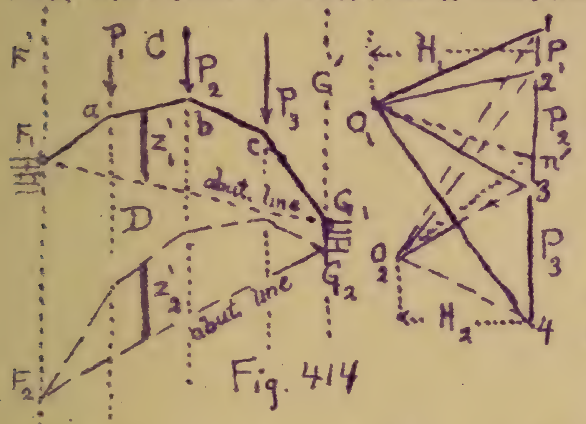


Fig. 414

Given a system of loads or vertical forces, P_1, P_2, etc Fig. 414, and two abutment verticals F′ and G′; if we lay off, vertically, to form a "load-line", 1...2 = P_1, 2...3 = P_2, etc., select any POLE, O_1, and join O_1...1, O_1...2, etc.; also, beginning at any point F_1 in the vertical F′, if we draw F_1...a ∥ to O_1..1 to intersect the line of P_1; then ab ∥ to O_1..2, and so on until finally a point G_1, in G′, is determined; then the figure F_1abcG_1 is an equilibrium polygon for the given loads and load verticals, and O_1...1234 is its "force diagram". The former is so called because the short segments F_1a, ab, etc, if considered to be rigid and imponderable rods, in a vertical plane, hinged to each other and the terminal ones to abutments F_1 and G_1, would be in equilibrium under the given loads hung at the joints. An infinite number of equilibrium polygons may be drawn for the given loads and abutment-verticals, by choosing different poles in the force diagram. [One other is shown in the figure formed of broken strokes; O_2 is its pole]. For all of these the following statements are true.

(1.) A line through the pole, ∥ to the abutment line cuts the load-line in the same point n′, whichever equilibrium polygon be used, (∴ any one will serve to determine n′)

(2.) If a vertical CD be drawn, giving an intercept z' in each of the equilibrium polygons, the product Hz' is the same constant for all the equilibrium polygons. That is (see Fig. 414) for any two of the polygons we have $H_1 : H_2 :: z'_2 : z'_1$; or $H_2 z'_2 = H_1 z'_1$.

(3.) The compression in each rod is given by that "ray" (in the force diagram) to which it is parallel.

(4.) The "pole distance", H, or ⊥ let fall from the pole upon the load-line, divides it into two parts which are the vertical components of the compressions in the abutment-rods respectively; H is the horizontal component of each, (and, in fact, of each of the compressions in all the other rods.) The compressions in the extreme rods may also be called the abutment reactions (oblique) and are given by the extreme rays.

(5.) THREE POINTS [no two in the same segment (or rod)] determine an equilibrium polygon for given loads. Having given then three points we may draw the equilibrium polygon by § 341.

375. SUMMATION OF PRODUCTS. Before proceeding to treat graphically any case of arch-ribs, a few processes in graphical arithmetic, as it may be called, must be presented, and thus established for future use

To make a summation of products of two factors in each by means of an equilibrium polygon.

CONSTRUCTION. Suppose it required to make the summation $\Sigma(xz)$ i.e., to sum the series

$$x_1 z_1 + x_2 z_2 + x_3 z_3 + \dots \text{ by graphics.}$$

Having first arranged the terms in the order of magnitude of the x's, we proceed as follows; suppos-

ing, for illustration, that two of the z's (z_3 and z_4) are negative (dotted in figure) See Fig. 416. These quantities x and z may be of any nature whatever, anything capable of being represented by a length, laid off to scale.

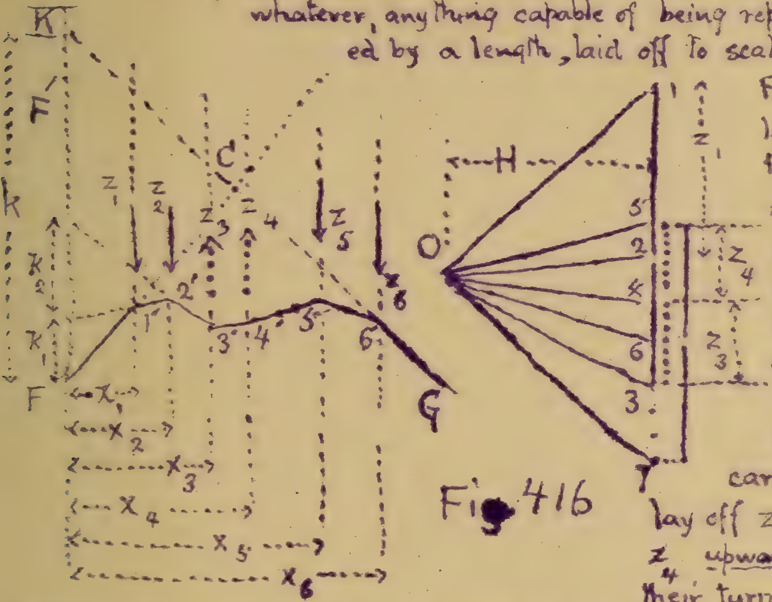


Fig. 416

First lay off the z's in their order on a vertical load-line taking care to lay off z_3 and z_4 upward in their turn.

Take any convenient pole O; draw the rays O...1, O...2, etc.; Then, having previously drawn vertical lines whose horizontal distances from an extreme left-hand vertical F' are made = x_1, x_2, x_3 etc. respectively, we begin at any point F, in the vertical F', and draw a line ‖ to O...1 to intersect the x_1 vertical in some point 1'; then 1'2' ‖ to O...2, and so on, following carefully the proper order. Produce the last segment (6'...G in this case) to intersect the vertical F' in some point K. Let KF = k (measured on the same scale as the x's), then the summation required is $\Sigma_1^n (xz) = Hk$ H is measured on the scale of the z's, which need not be the same as that of the x's.

[PROOF. From similar triangles $H : z_1 :: x_1 : k_1 \therefore x_1 z_1 = Hk_1$
and " " " $H : z_2 :: x_2 : k_2 \therefore x_2 z_2 = Hk_2$

and so on. But $H(k_1 + k_2 + \text{etc}) = H \times \overline{FK} = Hk$]

376. GRAVITY VERTICAL. From the same construction in Fig. 416 we can determine the line of action (or gravity vertical) of the resultant of the parallel vertical forces z_1, z_2, etc. (or loads); by prolonging the first and last segments to their intersection at C. The resultant of the system of forces or loads acts through C and is vertical in this case; its value being $= \Sigma(z)$, that is, it = the length 1....7 in the force diagram, interpreted by the proper scale. It is now supposed that the z's represent forces, the x's being their respective lever arms about F. If the z's represent the areas of small finite portions of a large plane figure, we may find a gravity-line (through C) of that figure by the above construction; each z being applied through the centre of gravity of its own portion

Calling the distance $\bar{x}$, between the verticals through C and F, we have also $\bar{x}.\Sigma(z) = \Sigma(xz)$ because $\Sigma(z)$ is the resultant of the ∥ z's. This is also evident from the proportion (similar triangles) $H : (1...7) :: \bar{x} : k$

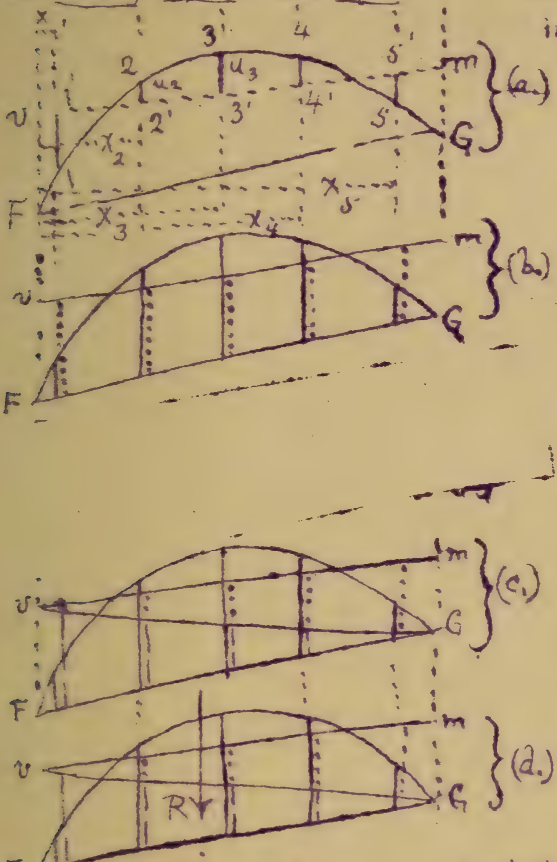
377. CONSTRUCTION for locating a line vm Fig. 417, at (a), in the polygon FG in such a position as to satisfy the two following conditions with reference to the vertical intercepts 1..1', 2..2', etc. between it and the given points 1, 2, 3, etc. of the perimeter of the polygon

COND. I. (Calling these intercepts u_1, u_2, etc, and their horizontal distances from a given vertical F', x_1, x_2, etc)

$\Sigma(u)$ is to $= 0$; i.e. the sum of the positive u's must be numerically = that of the negative (which here are at 1 and 5). An infinite number of positions of vm will satisfy condition I.

CONDITION (II.) $\Sigma(ux)$ is to $= 0$; i.e. the sum of the moments of the positive u's about F must = that

of the negative u's; i.e. that the moment of the resultant positive u's must = that of the resultant of the negative; and ∴ (Condit. I being already satisfied) these two resultants must be directly opposed and equal.



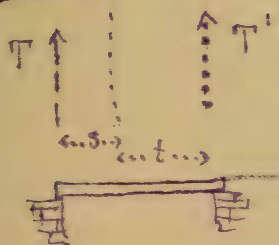
But the ordinates u in (a) are individually equal to the difference of the full and dotted ordinates in (b.), ∴ the conditions may be re written:

I. Σ (full ords. in (b.)) = Σ (dotted ords. in (b.))

II. Σ ([full in b.] x) = Σ ([dotted in b.] x)

i.e. the centres of gravity of the full and of the dotted in (b.) must lie in the same vertical.

Again by joining v G we may divide the dotted ordinates of (b) into two sets



which are dotted, and broken, respectively in (c.) Then, finally drawing in (d)

R, the resultant of full ords. of (c.)
T, the " " broken " " "
T', " " " dotted " " "

we are prepared to state in

still another and final form the conditions which vm must fulfil, viz.

(I.) $T + T'$ must $= R$; and (II.) The resultant of T and T' must act in the same vertical as R.

In short, the quantities T, T', and R must form a balanced system, considered as forces. All of which amounts practically to this: that if the verticals in which T and T' act are known and R be conceived as a load supported by a horizontal beam (see foot of Fig. 417, last figure) resting on piers in those verticals, then T and T' are the respective <u>reactions of those piers</u>. It will now be shown that the verticals of T and T' are easily found, <u>being independent</u> of the position of vm; and that both the vertical and the magnitude of being likewise independent of vm, are determined with facility in advance. For, if v be shifted up or down, all the broken ordinates in (c) or (d.) will change in the same proportion (viz. as vF changes), while the dotted ordinates, though shifted along their verticals, do not change in value; hence the shifting of v affects neither the vertical nor the value of T', nor the vertical of T. The <u>value</u> of T, however, is proportional to vF. Similarly, if m be shifted, up or down, T' will vary proportionally to mG but its vertical, or line of action, remains the same. T is unaffected in any way by the shifting of m. R, depending for its value and position on the full ordinates of (c.) Fig. 417, is independent of the location of vm. We may ∴ proceed as follows

1st. Determine R graphically, in amount and position, by means of § 376

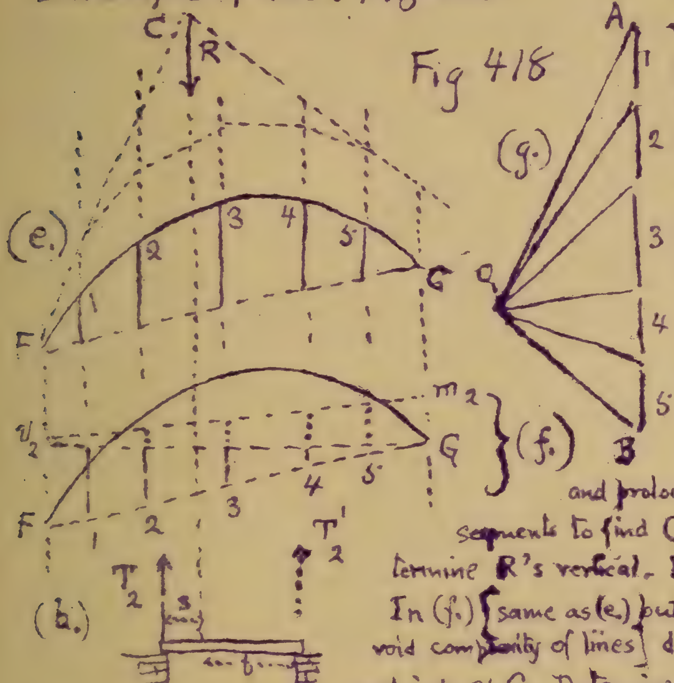
2ndly Determine the <u>verticals</u> of T and T' by <u>any trial</u> position of vm (call it v_2m_2), and the corresponding trial values of T and T' (call them T_2 and T'_2)

3rdly By the fiction of the horizontal beam, construct or compute the true values of T and T', and then determine the

true distances vF and mG by the proportions

$$vF : v_2F :: T : T_2 \quad \text{and} \quad mG : m_2G :: T' : T'_2.$$

Example of this. Fig 418.



From A toward B in (g.) lay off the lengths (or lines proportional to them) of the full ordinates 1, 2, etc of (e.) Take any pole O, and draw the dotted equil. polygon of (e.), and prolong its extreme segments to find C and thus determine R's vertical. $R = \overline{AB}$.

In (f.) [same as (e.) but shifted, to avoid complexity of lines] draw a trial v_2m_2 and join v_2G. Determine the sum T_2 of the broken ordinates (between v_2G and FG) and its vertical line of application, precisely as in dealing with R; also T'_2 that of the dotted ordinates (five) and its vertical. Now the true T = ~~xxx~~ $= Rt \div (s+t)$ and the true $T' = Rs \div (s+t)$

Hence compute $\overline{vF} = (T \div T_2)\,\overline{v_2F}$ and $\overline{mG} = (T' \div T'_2)\,\overline{m_2G}$

and by laying them off vertically upward from F and G respectively we determine v and m, i.e. the line vm to fulfil the conditions imposed at the beginning of this article, relating to the vertical ordinates intercepted between vm and given points on the perimeter of a polygon or curve.

378 CLASSIFICATION OF ARCH-RIBS, or ELASTIC ARCHES, according to continuity and modes of support. In the accompanying figures the full curves show the unstrained form of the rib (before any load, even its own weight, is permitted to come upon it; the dotted curve shows its shape (much exaggerated) when bearing a load. For a given loading THREE CONDITIONS must be given to determine the special equilibrium polygon (§§ 366 and 367).

CLASS A. Continuous rib, free to slip laterally on the piers, which have smooth horizontal surfaces Fig. 420

This is chiefly of theoretic interest; its consideration will be taken up after the others. The pier reactions are necessarily vertical, just as if it were a straight horizontal beam.

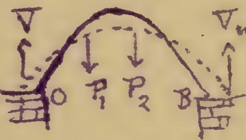
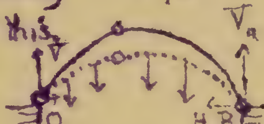


Fig. 420

CLASS B. Rib of THREE HINGES, two at the piers, and one intermediate (usually at the crown); Fig. 36 is an example of this. That is, the rib is discontinuous and of two segments. Since at each hinge the moment of the stress couple must be zero, the special equilibrium polygon must pass through the hinges. Hence as three points fully determine an equilibrium polygon for given loads, the special equilibrium is drawn by § 341. See Fig. 421

Fig. 421



CLASS C. Rib of TWO HINGES, these being at the piers; the rib continuous between. The piers are considered immovable, i.e. the span cannot change as a consequence of loading. It is also considered that the rib is fitted to its hinges at a definite temperature, and is then under no constraint from the piers (as if it lay flat on the ground) not even its own weight being permitted to

act when it is finally put in position. When the "false works" or temporary supports are removed, stresses are induced in the rib both by its loading, including its own weight, and by a change of temperature. Stresses due to temperature may be ascertained separately and then combined with those due to the loading. [Classes A and B are not subject to temperature stresses] Fig. 422 shows a rib of two hinges, at ends.

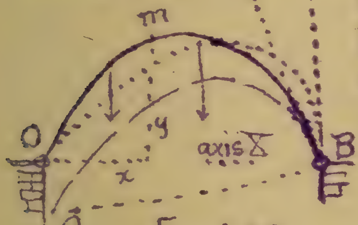


Fig. 422

Conceive the dotted curve (form and position under strain) to be superposed on the continuous curve (form before strain) in such a way that B and its tangent-line (which has been displaced from its original position) may occupy their previous position. This gives us the broken curve O_nB. OO_n is ∴ O's displacement relatively to B and B's tangent. Now the piers being immovable $\overline{O_nB}$ (right line) = $\overline{OB}$; i.e. the X projection (or Δx) of OO_n upon OB (taken as an axis of X) is zero compared with its Δy. Hence as one condition to fix the special equilibrium polygon for a given loading we have (from § 373)

$$\int_0^B [My\,ds \div EI] = 0 \ldots (1)$$

The other two are that the special equilibrium pol. } must pass thro' O ... (2)
" " " B .. (3)

CLASS D. Rib with FIXED ENDS and no hinges; i.e. continuous. Piers immovable. The ends may be fixed by being inserted, or built, in the masonry, or by being fastened to large plates which are bolted to the piers. [The St. Louis Bridge and that at Coblenz over the Rhine are of this class] Fig. 423. In this class there being no hinges we have no point given in advance through which the special equilibrium polygon must pass. However, since O's displacement relatively (and absolutely) to B and B's tangent is zero,

both Δx and Δy [see § 373] = zero. Also, the tangent-lines both at C and B being fixed in direction, the angle between them is the same under loading, or change of temperature, as when the rib was first placed in position under no strain and at a definite temperature.

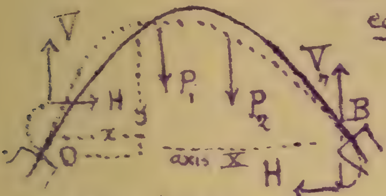


Fig. 423

Nence the conditions for locating the special equilibrium polygon are

$$\int_0^B \frac{M\,ds}{EI} = 0\,; \qquad \int_0^B \frac{My\,ds}{EI} = 0\,; \qquad \int_0^B \frac{Mx\,ds}{EI} = 0.$$

In the figure the imaginary rigid prolongations at the ends are shown [See § 366]

EXAMPLE OF CLASS C. Prof. Eddy's Method; see p. 199

379. ARCH RIB HINGED AT THE ENDS; i.e. with two hinges. Location of the special equilibrium polygon. We here suppose the rib homogeneous (i.e. the modulus of Elasticity E is the same throughout) that it is a "curved prism" (i.e. that the moment of inertia I of the cross section is constant) that the piers are on a level, and that the rib curve is symmetrical about a vertical line. Fig. 424

For each point m of the rib curve we have an x and y (both known, being the co-ordinates of the point), and also a z (intercept between rib and special equilibrium polygon) and a z' (intercept between the spec. eq. pol. and the axis X (which is OB)

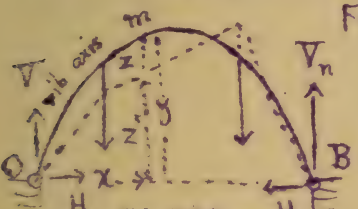


Fig. 424

The third condition given in § 378 for Class C may be transformed as follows, remembering [§ 367 eq.(1)] that $\underline{M} = Hz$ at any point m of the rib;

(and that EI is constant):

$$\frac{1}{EI}\int_0^B Myds = 0, \text{ i.e. } \frac{H}{EI}\int_0^B zyds = 0 \quad \therefore \int_0^B zyds = 0$$

but $z = y - z'$ $\Big\}$ $\therefore \int_0^B (y-z')yds = 0$ i.e. $\int_0^B yyds = \int_0^B yz'ds$ ....(1.)

In practical graphics we can not deal with infinitesimals; hence we must substitute Δs a small finite portion of the rib-curve for ds. Eq. (1) now reads $\sum_0^B yy\Delta s = \sum_0^B yz'\Delta s$

But if we take ALL THE Δs's EQUAL, Δs is a common factor and cancels out, leaving as a final form for eq. (1)

$$\sum_0^B (yy) = \sum_0^B (yz') \;\ldots\ldots \qquad (1)'$$

The other two conditions are that the spec. eq. pol. begins at O and ends at B. (The subdivision of the rib-curve into EQUAL Δs's will be observed in all problems henceforth.)

DETAIL OF THE CONSTRUCTION. Given the arch-rib O B, Fig. 425, with specified loading. Divide the curve into eight equal Δs's and draw a vertical through the middle of each. Let the loads borne by the respective Δs's be P_1, P_2, etc. and with them form a vertical load line AC to some convenient scale. With any convenient pole O" draw

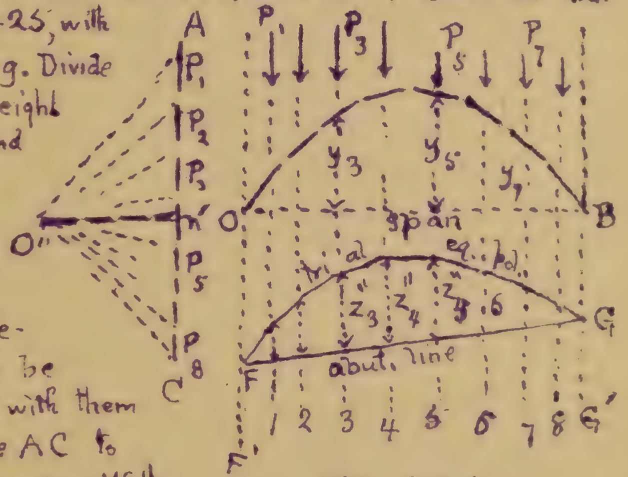
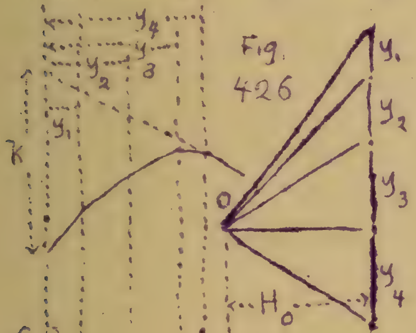


Fig. 425

a trial force diagram $O''AC$, and a corresponding trial equilibrium polygon FG, beginning at any point in the vertical F'. Its ordinates z''_1, z''_2, etc. are proportional to those of the special equil. pol. sought (whose abutment line is OB) [§ 374 a (2)]. We next use it to determine π' [see § 374 a]. We know that OB is the "abutment" line of the required special polygon, and that ∴ its pole must lie on a horizontal through π'. It remains to determine its H, or pole distance, by equation (1)' just given, viz.: $\Sigma_1^8 yy = \Sigma_1^8 yz'$

First by § 375 find the value of the summation $\Sigma_1^8 (yy)$ which from symmetry we may write $= 2\Sigma_1^4 (yy) = 2[y_1 y_1 + y_2 y_2 + y_3 y_3 + y_4 y_4]$



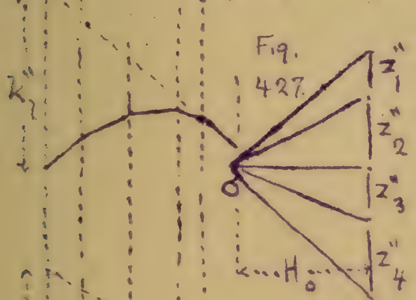
Hence, Fig. 426, we obtain

$$\Sigma_1^8 (yy) = 2[H_0 k]$$

Next, also by § 375, see Fig 427, using the same pole distance H_0 as in Fig. 426, we find

$$\Sigma_1^4 (yz'') = H_0 k''_l \; ; \text{ i.e.}$$

$$y_1 z''_1 + y_2 z''_2 + y_3 z''_3 + y_4 z''_4 = H_0 k''_l .$$



Again, since $\Sigma_5^8 (yz'') =$
$y_8 z''_8 + y_7 z''_7 + y_6 z''_6 + y_5 z''_5$
which from symmetry (of rib)
$= y_1 z''_8 + y_2 z''_7 + y_3 z''_6 + y_4 z''_5$
we obtain from Fig. 428

$$\Sigma_5^8 (yz'') = H_0 k''_r$$

(same H_0) and ∴

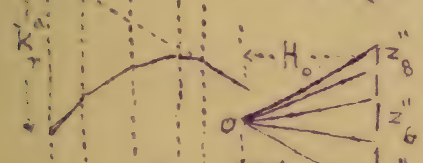


Fig. 428

$\Sigma_1^8 (yz'') = H_0 (k_l'' + k_r)$ If now we find that $k_l'' + k_r'' = 2k$, the condition $\Sigma_1^8 (yy) = \Sigma_1^8 (yz'')$ is satisfied, and the pole distance of our trial polygon in Fig. 425, is also that of the special polygon sought; i.e. the z'''s are identical in value with the z''s of Fig. 424. In general of course we do not find that $k_l'' + k_r'' = 2k$. Hence the z'''s must all be increased in the ratio $2k : (k_l'' + k_r'')$ to become equal to the z''s. That is, the pole distance H of the spec. equil. polygon must be

$$H = \frac{k_l'' + k_r''}{2k} H''$$

(in which H'' = the pole distance of the trial polygon) since from §339 the ordinates of two equilibrium polygons (for the same loads) are inversely as their pole distances. Having thus found the H of the special polygon, knowing that the pole must lie on the horizontal thro' n'', Fig. 425, it is easily drawn, beginning at O. As a check, it should pass through B.

For its utility see §367, but it is to be remembered that the stresses as thus found in the different parts of the rib under a given loading, must afterwards be combined with those resulting from change of temperature and the shortening of the rib axis due to the tangential thrusts, before the actual stress can be declared in any part.

380. ARCH RIB OF FIXED ENDS and no hinges. EXAMPLE OF CLASS D. As before E and I are constant along the rib. Piers immovable. Rib curve symmetrical about a vertical line. Fig. 429 shows such a rib under any loading. Its span is OB, which is taken as an axis X. The co-ordinates of any point m' of the rib curve are x and y, and z is the vertical intercept between m' and the special equilibrium polygon (as yet un-

known, but to be constructed) Prof. Eddy's method will now be given for finding the special equil. polygon. The ~~three~~ conditions it must satisfy (see § ~~367~~ 378 Class D, remembering that E and I are constant and that $M = Hz$ from § 367) are

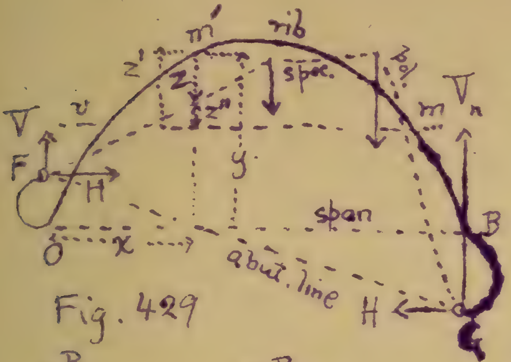


Fig. 429

$$\int_0^B z\,ds = 0; \quad \int_0^B xz\,ds = 0; \quad \text{and} \int_0^B yz\,ds = 0 \quad \ldots.(1.)$$

Now suppose the auxiliary reference line (straight) vm to have been drawn satisfying the requirements, with respect to the rib curve that

eqs. (2) ...... $\int_0^B z'ds = 0$ and $\int_0^B xz'ds = 0$

in which z' is the vertical distance of any point m' from vm and x the abscissa of m' from O.

From Fig. 429, letting z'' denote the vertical intercept (corresponding to any m') between the spec. polygon and the auxiliary line vm, we have $z = z' - z''$, hence the three conditions in (1.) become

$$\int_0^B (z'-z'')ds = 0 \quad \text{i.e., see eq.s (2),} \quad \int_0^B z''ds = 0 \quad (3.)$$

$$\int_0^B x(z'-z'')ds = 0; \quad " \quad " \quad " \quad " \quad \int_0^B xz''ds = 0 \quad (4)$$

and $\int_0^B y(z'-z'')ds = 0$; ∴ by transposition $\int_0^B yz'ds = \int_0^B yz''ds$ ..(5) <u>provided</u>

vm has been located as prescribed.

For graphical purposes, having subdivided the rib curve into a number of small EQUAL Δs's, and drawn a vertical through the middle of each, we first by § 377 locate vm to satisfy the conditions

$$\Sigma_0^B (z') = 0 \quad \text{and} \quad \Sigma_0^B (xz') = 0 \qquad (6)$$

(see eq.s (2); the Δs cancels out); and then locate the special equil. polygon, with vm as a reference-line, by making it satisfy the conditions

$$\Sigma_0^B (z'') = 0 \ldots (7); \quad \Sigma_0^B (xz'') = 0 \ldots (8); \quad \Sigma_0^B (yz'') = \Sigma_0^B (yz') \ldots (9)$$

(obtained from eqs. (3), (4), (5) by putting $ds = \Delta s$, and canceling.)

Conditions (7) and (8) may be satisfied by an infinite number of polygons drawn to the given loading. Any one of these being drawn, as a trial polygon, we determine for it the value of the sum $\Sigma_0^B (yz'')$ by § 375, and compare it with the value of the sum $\Sigma_0^B (yz')$ which is independent of the special polygon and is obtained by § ~~377~~, 375. [N.B. It must be understood that the quantities (lines) x, y, z, z', and z'' here dealt with are those pertaining to the verticals drawn through the middles of the respective Δs's. see Fig. 429] If these sums are not equal, the pole distance of the trial equil. polygon must be altered in the proper ratio (and thus change the z'''s in the inverse ratio) necessary to make these sums equal and thus satisfy condition (9). The alteration of the z'''s, all in the same ratio, will not interfere with conditions (7) and (8) which are already satisfied

381. *DETAIL OF CONSTRUCTION* of last paragraph. ARCH-RIB OF FIXED ENDS. *As* an example take a span of the St Louis Bridge with live load covering the half span on the left. Fig. 430, where the vertical scale

is much exaggerated for the sake of distinctness. Divide into eight equal Δs's. P_1, etc. are the loads coming upon the respective Δs's

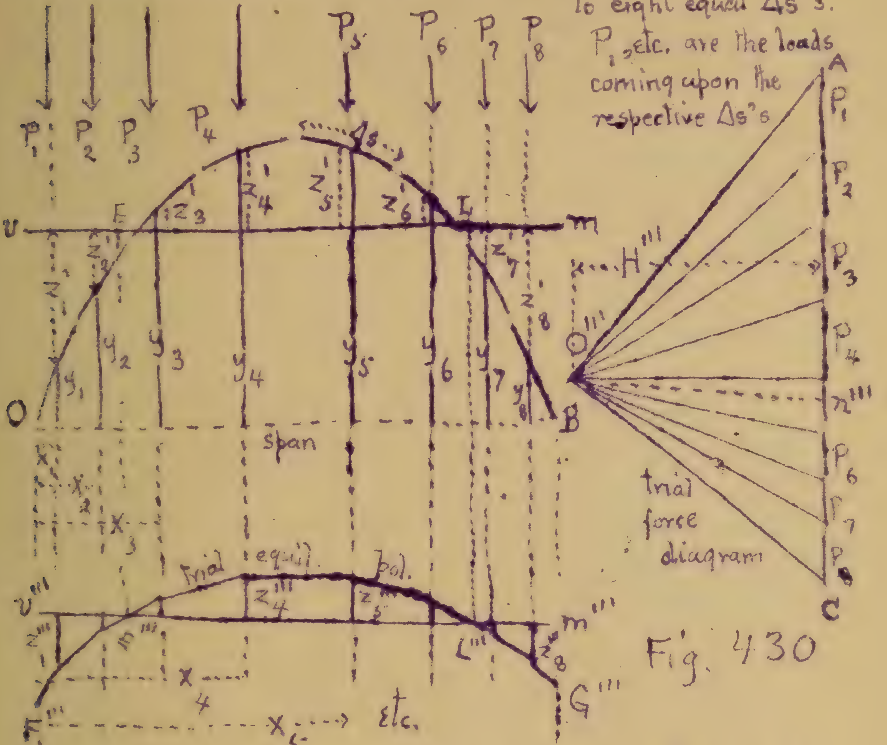


Fig. 430

First to locate vm, by eqs. (6.); from symmetry it must be horizontal. Draw a trial vm (not shown in the figure) and if the $(+z')$'s exceed the $(-z')$'s by an amount z'_0, the true vm will lie a height $\frac{1}{n} z'_0$ above the trial vm (or below, if vice versâ). n = the number of Δs's.

Now lay off the load-line AC, on the right, (to scale), take any convenient trial pole O''' and draw a corresponding trial equil. polygon $F'''G'''$. In $F'''G'''$, by § 377, locate a straight line $v'''m'''$ so as to make $\Sigma_1^8 (z''') = 0$ and $\Sigma_1^8 (xz''') = 0$

[We might now redraw $F'''G'''$ in such a way as to bring $v'''m'''$ into a horizontal position, thus: first determine a point n''' on the load-line by drawing $O'''n'''$ ∥ to $v'''m'''$

take a new pole on a horizontal through n''', with the same H''', and draw a corresponding equil. polygon; in the latter $v'''n'''$ would be horizontal. We might also shift this new trial polygon upwards so as to make $v'''m'''$ and vm co-incide. It would satisfy conditions (7) and (8), having the same z''''s as the first trial polygon; but to satisfy condit. (9) it must have its z''''s altered in a certain ratio, which we must now find. But we can deal with the individual z''''s just as well in their present positions in Fig. 430.] The points E and L in vm, vertically over E''' and L''' in $v'''m'''$, are now fixed; they are the intersections of the spec. pol. required with vm.

The ordinates between $v'''m'''$ and the trial equilibrium polygon have been called z''' instead of z''; they are proportional to the respective z'''s of the required special polygon.

The next step is to find in what ratio the (z''')'s need to be altered (or H''' altered in inverse ratio) in order to become the (z'')'s; i.e. in order to fulfil condition (9) viz.

$$\sum_1^8 (yz'') = \sum_1^8 (yz')$$

This may be done precisely as for the rib with two hinges, but the negative (z''')'s must be properly considered. (§ 375) See Fig. 431 for the detail. Negative z''s or z''''s are dotted

From Fig. 431 a

$$\sum_1^4 (yz') = H_0 k$$

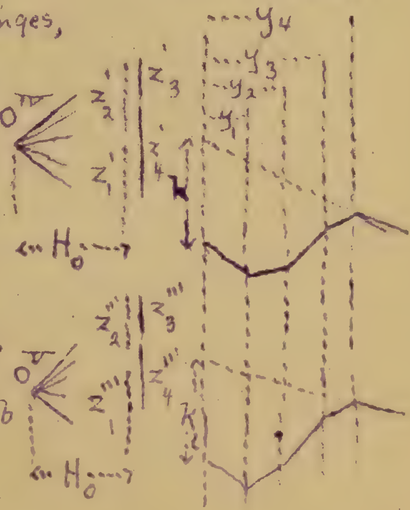
∴ from symmetry

$$\sum_1^8 (yz') = 2 H_0 k$$

From Fig. 431 b we

Fig 431 a

Fig. 431 b



have $\Sigma_1^4(yz''') = H_0 k_r$

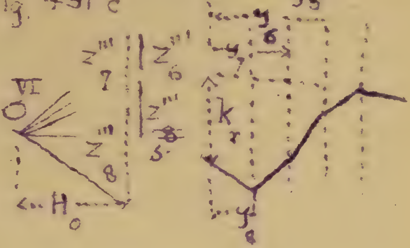
and from Fig. 431 c

$\Sigma_5^8(yz''') = H_0 k_l$

[The same pole distance H_0 is taken in all these constructions]

$\therefore \Sigma_1^8(yz''') = H_0(k_l + k_r)$

Fig. 431 c



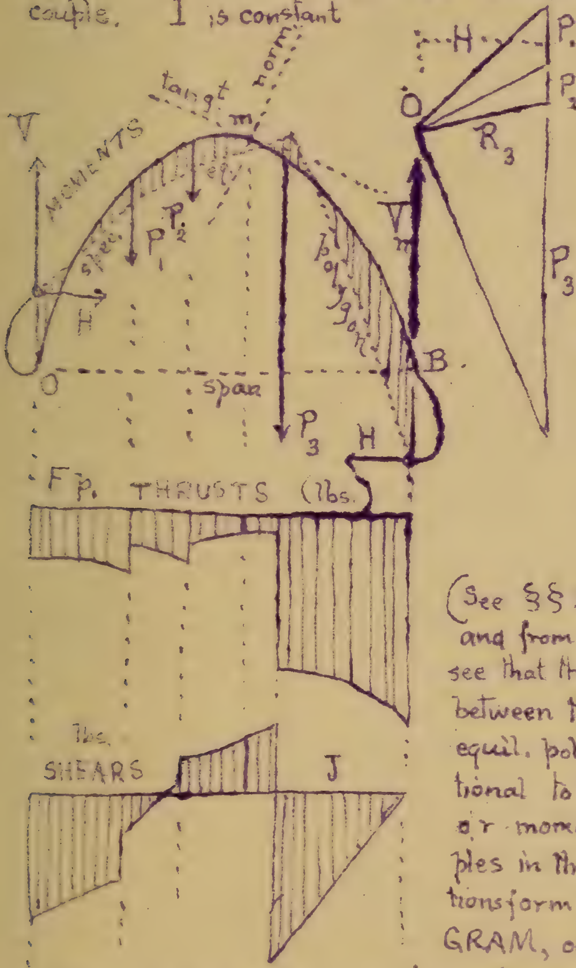
If, then, $H_0(k_l + k_r) = 2H_0 k$ condition (9) is satisfied by the z''''s. If not, the true pole distance for the special equil. polygon of Fig. 430 will be

$$H = \frac{k_l + k_r}{2k} \cdot H'''$$

With this pole distance and a pole in the horizontal thro' n''' (Fig. 430) the force diagram may be completed for the required special polygon, and this latter may be drawn by beginning at the point E, in vm, and through it draw a segment ‖ to the proper ray of the force diagram. In our present figure (430) this "proper ray" would be the ray joining the pole with the point of meeting of P_2 and P_3 on the load-line. Having this one segment of the special polygon the others are added in an obvious manner, and thus the whole polygon completed. It should pass thro' L, but not O and B.

For another loading a different special equil. polygon would result, and in each case we may obtain the thrust, shear, and moment of stress couple for any cross-section of the rib, by § 367. To the stresses computed from these, should be added (algebraically) those occasioned by change of temperature and by shortening of the rib as occasioned by the thrust along the rib. These "temperature stresses", and stresses due to rib-shortening, will be considered in a subsequent paragraph.

382 STRESS DIAGRAMS. Take an arch-rib of Class D, § 378, i.e. of fixed ends and suppose that for a given loading (including its own weight) the special equil. polygon and its force diagram have been drawn [§ 381]. It is required to indicate graphically the variation of the three stress-elements for any section of the rib, viz. the thrust, shear, and mom. of stress-couple. I is constant.



If at any point m of the rib a section is made, then the stresses in that section are classified into three sets (Fig. 432)

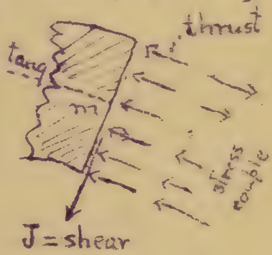


Fig. 432

(See §§ 295 and 367) and from § 367 eq. (3) we see that the vertical intercept between the rib and the special equil. polygon being proportional to the ~~products~~ Hz or moments of the stress-couples in the corresponding sections form a MOMENT DIAGRAM, on inspection of which

Fig. 433.

we can trace the change in this moment $Hz = \frac{p_2 I}{e}$, (and hence the variation of

the stress per square inch, p_2, in the outermost fibre of any section (tension or compression) at distance e from the gravity axis of the section) from section to section along the rib.

By drawing a tangent and normal at any point m of the rib axis [see Fig 433] and projecting upon them, in turn, the proper ray (R_3 in Fig. 433) (see eqs. 1 and 2 of § 367) we obtain the values of the thrust and shear for the section at m. When found in this way for a number of points along the rib their values may be laid off as vertical lines from a horizontal axis, in the verticals containing the respective points, and thus a THRUST DIAGRAM and a SHEAR DIAGRAM may be formed, as constructed in Fig. 433. Notice that where the moment is a maximum or minimum the shear passes through the value zero (compare § 240), either gradually or suddenly, according as the max. or min. occurs between two loads or in passing a load.

Also it is evident, from the geometrical relations involved that at those points of the rib where the tangent-line is parallel to the segment of the equil. polygon just below, or above, the thrust is a maximum (a local maximum) the moment (of stress couple) is either a max. or a minimum and the shear is zero

From the moment $Hz = \frac{p_2 I}{e}$, $p_2 = \frac{Hze}{I}$

may be computed. From the thrust $= F p_1$, $p_1 = \frac{\text{thrust}}{F}$.
(F = area of cross-section)
may be computed. Hence the greatest compression per sq. inch $(p_1 + p_2)$ may be found in each section. A separate stress diagram might be constructed for this quantity $(p_1 + p_2)$ (after adding temper. stress, etc.)
Its max. value, wherever it occur in the rib, must be made safe by proper designing of the rib. The max. Shear J_m can be used as in § 256 to determine thickness of web, if the section is I shaped.

383. TEMPERATURE STRESSES. In an ordinary bridge truss, and straight horizontal girders free to expand or contract longitudinally, and in Classes A and B of § 378 of arch-ribs, there are no stresses induced by change of temperature; for the form of the beam or truss is under no constraint from the manner of support; but with the arch rib of two hinges (hinged ends, Class C) and of fixed ends (Class D) having immovable piers which constrain the distance between the two ends to remain the same at all temperatures, stresses called temperature stresses are induced in the rib whenever the temperature, t, is not the same as that, t_0, when the rib was put in place. These may be determined, as follows, as if they were the only ones, and then combined, algebraically, with those due to the loading.

384. TEMPERATURE STRESSES IN THE ARCH RIB OF HINGED ENDS (Class C § 378) Fig. 434.

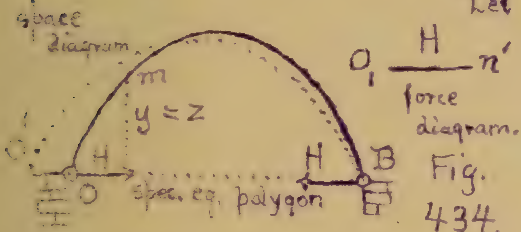


Fig. 434.

Let E and I be constant with other postulates as in § 379. Let t_0 = temperature of erection, and t = any other temperature, also l = length of span = OB (invariable) and η = coefficient of expansion of the material of the curved beam or rib (see § 199) at temperature t there must be a horizontal reaction H at each hinge to prevent expansion into the form $O'B$ (dotted curve) which is the form natural to the rib for temperature t and without constraint. We may consider the actual form OB as having resulted from the unstrained form $O'B$ by displacing O' to O, i.e. producing a horizontal displacement $O'O = l(t-t_0)\eta$

But $OO = \Delta x$ (see §§ 373 and 374); (N.B. B's tangent has moved, but this does not affect Δx if the axis X is horizontal, as here, co-inciding with the span); and the ordinate y of any point m of the rib is identical with its z or intercept between it and the spec. equil. polygon, which here consists of one segment only, viz. OB. Its force diagram consists of a single ray $O_1 n'$; see Fig. 434. Now (§ 373)

$$\Delta x = \frac{1}{EI}\int_0^B My\,ds\text{; and } M = Hz \text{ = in this case } Hy$$

$$\therefore \quad l(t-t_0)\eta = \frac{H}{EI}\int_0^B y^2 ds\text{;}$$ hence for graphics, and EQUAL Δs's, we have

$$EI\,l(t-t_0)\eta = H\,\Delta s\,\Sigma_0^B y^2 \quad \text{. . . . (1)}$$

From eq. (1) we determine H, having divided the rib-curve into from eight to twenty equal parts each called Δs. For instance, for wrought iron, t and t_0 being expressed in Fahrenheit degrees, $\eta = 0.0000066$. If E is expressed in lbs. per square inch, all linear quantities should be in inches and H will be obtained in pounds.

$\Sigma_0^B y^2$ may be obtained by § 375, or may be computed. H being known, we find the moment of stress couple $= Hy$, at any section, while the thrust and shear at that section are the projections of H, i.e. of $O_1 n'$, upon the tangent and normal. The stresses due to these may then be determined in any section, as already so frequently explained, and then combined with those due to loading.

385. TEMPERATURE STRESSES IN THE ARCH-RIB WITH FIXED ENDS. See Fig. 435. Same postulates as to symmetry, E constant, etc. as in § 380. t and t_0

as before. Here, as before, we consider the rib to have reached its actual form under temperature t by having ~~had~~ its span forcibly shortened from the length natural to temp. t, viz. $O'B$, to the actual length OB which the immovable piers compel it to assume. But here, since the tangents at O and B are to be the same in direction, under constraint as before, the two forces H, representing the action of the piers on the rib, must act on imaginary rigid prolongations at an unknown distance d above the span OB. To find H and d we need two equations. From § 373 we have, since $M = Hz = H(y-d)$,

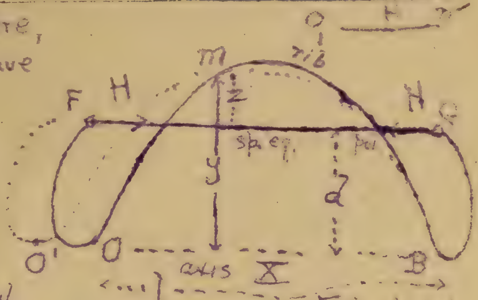


Fig. 435

$$\Delta x \text{ i.e. } O'O \text{, i.e. } l(t-t_0)\eta = \frac{H}{EI}\int_0^B (y-d)\,y\,ds \quad \ldots(2)$$

or, graphically, with equal Δs's,

$$EI\,l(t-t_0)\eta = H\Delta s\left[\sum_0^B y^2 - d\sum_0^B y\right] \quad \ldots(3)$$

Also, ~~since~~ there has been no change in the angle between end-tangents, we must have from § 374

$$\frac{1}{EI}\int_0^B M\,ds = 0 \text{ i.e. } \frac{H}{EI}\int_0^B z\,ds = 0\text{; i.e. } \int_0^B (y-d)\,ds = 0$$

or, for graphics, with equal Δs's, $\sum_0^B y = nd \quad \ldots(4)$

in which n denotes the number of Δs's. From (4) we determine d, and then from (3) can compute H. Drawing the horizontal FG, it is the spec. equil. polygon, and the moment of the stress couple at any section $= Hz$, while the thrust and shear are the projections of $H = Om'$ on the tangent and normal respectively.

For example, in one span, of 550 feet, of the St. Louis Bridge, having a rise of 55 feet and fixed at the ends, the force H of Fig. 435 is = 108 tons when the temperature is 80° Fahr. higher than the temp. of erection, and the enforced span is $3\frac{1}{4}$ inches shorter than the distance O'B the span natural to that higher temperature. Evidently, if the actual temperature t is lower than that, t_0, of erection, H must act in a direction opposite to that of Figs. 435 and 434.

386. STRESSES DUE TO RIB-SHORTENING. In § 369, Fig. 407, the shortening of the element AE to a length A'E, due to the uniformly distributed thrust, $p_1 F$, was neglected as an agency causing a change of curvature and form in the rib axis. This change in the length of the different portions of the rib curve, may be treated as if it were due to a change of temperature. *For example*, from § 199 we see that a thrust of 50 tons coming upon a sectional area of F = 10 sq. inches in an iron rib, whose material has a modulus of Elasticity = E = 30 000 000 lbs. per sq. inch, and a coefficient of expansion η = .0000066 per degree Fahrenheit, produces a shortening equal to that due to a fall of temperature $(t_0 - t)$ derived as follows: see § 199

$$(t - t_0) = \frac{P}{F E \eta} = \frac{100\,000 \text{ lbs}}{10 \times 30\,000\,000 \times .0000066} = 50^\circ$$

Fahrenheit (units, inch and lb.).

Practically, then, since most metal arch bridges of classes C and D are rather flat in curvature, and the thrust due to ordinary modes of loading does not vary more than 20 or 30 per cent. along the rib, an imaginary fall of temperature corresponding to an average thrust in any one case of loading may be made the basis of a

construction similar to that in § 384 or § 385 (according as the ends are hinged, or fixed) from which new thrusts, shears, and stress-couple moments, may be derived to be combined with those previously obtained for loading and for change of temperature.

387. RESUMÉ. It is now seen how the stresses per square inch, both shearing and compression (or tension) may be obtained in all parts of any section of a solid arch-rib or curved beam of the kinds described, by combining the results due to the three separate causes, viz.: the load, change of temperature, and rib-shortening caused by the thrusts due to the load. That is, in any cross section, the stress in the outer fibre is, [letting T'_h T''_h T'''_h denote the thrusts due to the three causes, respectively, above mentioned; $(Hz)'$ $(Hz)''$ $(Hz)'''$ the moments)

$$= \frac{T'_h + T''_h + T'''_h}{F} + \frac{e}{I}\left[(Hz)' + (Hz)'' + (Hz)'''\right] \quad \ldots (1)$$

i.e., lbs. per sq. inch compression (if those units are used)

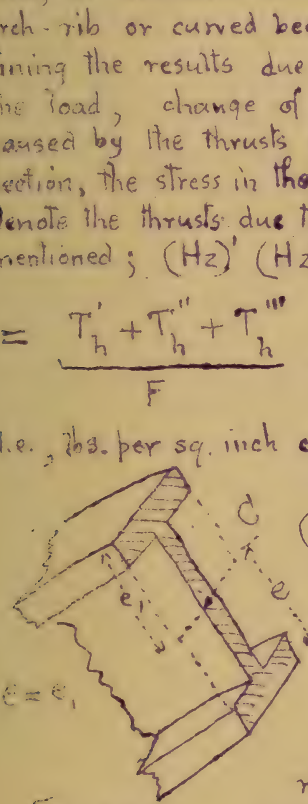


Fig 436.

Fig. 436 shows the meaning of e (the same used heretofore) I is the moment of inertia of the section about the gravity axis (horizontal) C. F = area of cross-section. [$e_1 = e$ cr. section symmet about C]

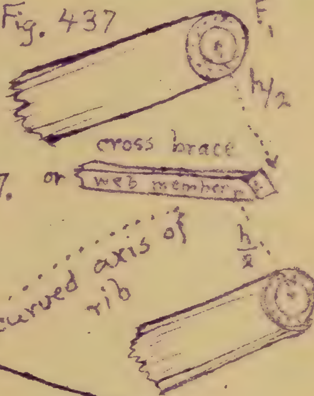
For a given loading we may find the maximum stress in a given rib, or design the rib so that this maximum stress shall be safe for the material employed. Similarly, the resultant shear (total not per sq. inch) = $J' + J'' + J'''$ is obtained for any section To compute a proper

thickness of web, spacing of rivets, etc.

387. THE ARCH-TRUSS, or braced arch. An open-work truss, if of homogeneous design from end to end, may be treated as a beam of constant section and constant moment of inertia, and if curved, like the St. Louis Bridge and the Coblenz Bridge (see § 378 Class D), may be treated as an arch-rib. Its moment of inertia consists of the value

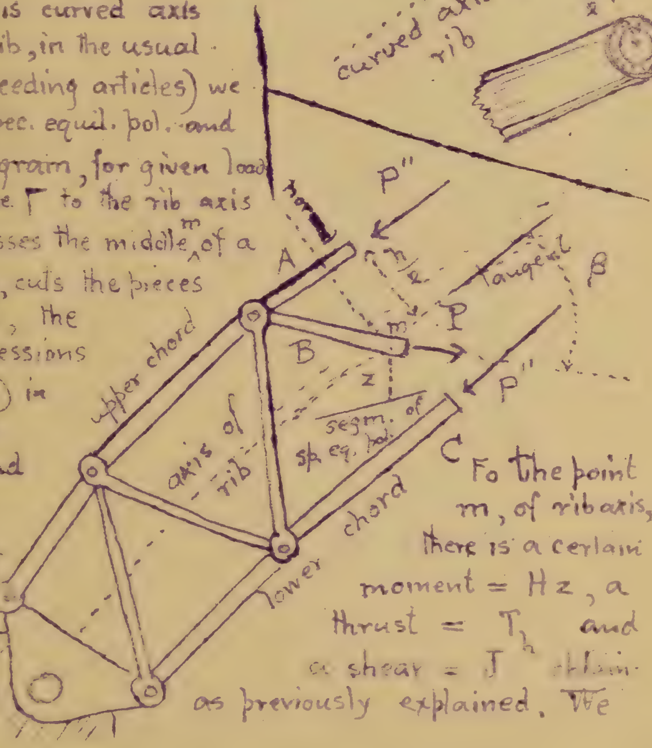
$$I = 2F_1\left(\frac{h}{2}\right)^2$$

where



F_1 is the sectional area of one of the pieces ∥ to the curved axis midway between them, Fig. 437, and h = distance between them.

Treating this curved axis as an arch-rib, in the usual way (see preceding articles) we obtain the spec. equil. pol. and its force diagram, for given loading. Any plane ⊥ to the rib axis where it crosses the middle m of a web member, cuts the pieces A, B, and C, the total compressions (or tensions) in which are thus found



To the point m, of rib axis, there is a certain moment = Hz, a thrust = T_h and a shear = J obtain as previously explained. We

may then write $P \sin \beta = J$, and thus determine whether P is a tension or compression; then putting $P' + P'' \pm P \cos \beta = T_h$ (in which P is taken with a plus sign if a compression and minus if tension) and $(P' - P'')\frac{h}{2} = Hz$ ...(3)

we compute P' and P'', which are assumed to be both compressions here. β is the angle between the web member and the tangent to rib-axis at m, the middle of the piece. See Fig. 406, as an explanation of the method just adopted.

HORIZONTAL STRAIGHT GIRDERS.

388. ENDS FREE TO TURN. This corresponds to an arch-rib with hinged ends but it must be understood that there is no hindrance to horizontal motion. Fig. 439

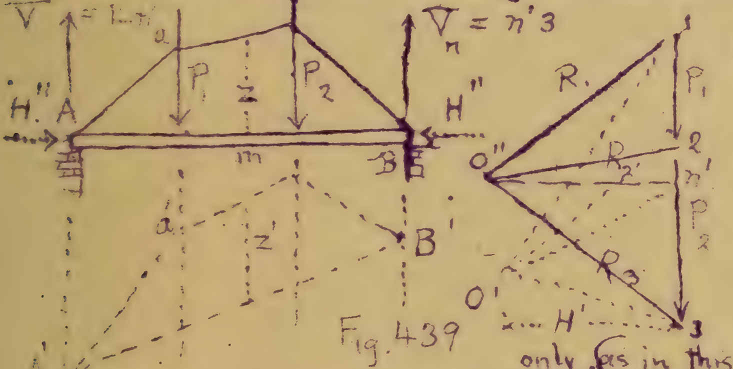


Fig. 439

In treating a straight beam, slightly bent under vertical forces only, (as in this case with no horizontal constraint) as a particular case of an arch rib, it is evident that since the pole distance must be zero the special equil. polygon will have all its segments vertical, and the corresponding force diagram reduces to a single vertical line (the load line). The first and last segments must pass through A and B (points of no moment) respectively, but being vertical will not intersect P_1 and P_2; i.e. the remainder of the special equilibrium polygon lies at an infinite distance above the

span AB. Hence the actual spec. equil. pol. is useless.
However, knowing that the shear, J, and the moment M (of stress couple) are the only quantities pertaining to any section m (Fig. 439) which we wish to determine, (since there is no thrust along the beam) and knowing that an imaginary force H'' applied horizontally at each end of the beam would have no influence in determining the shear and moment at m as due to the new system of forces thus formed, hence the shears and moments may be obtained graphically from this new system (viz.: the loads P_1, etc., the vertical reactions V and V_n, and the two equal and opposite H'''s).
[Evidently, since H'' has no moment about the neutral axis, (or gravity axis here), of m, the moment at m will be unaffected by it; and since H'' has no component ⊥ to the beam at m, the shear at m is the same in the new system of forces, as in the old, before the introduction of the H'''s.]

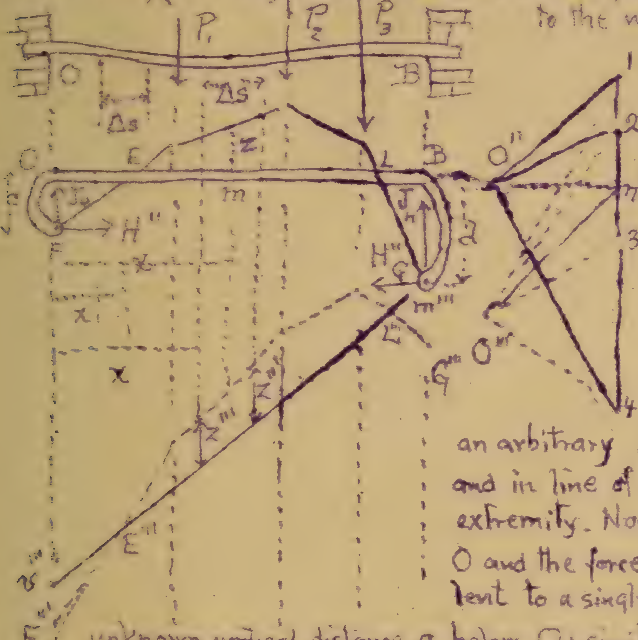
Hence lay off the load-line 1..2..3, Fig. 439, and construct an equil. polyg. which shall pass through A and B and have any convenient arbitrary H'' (force) as a pole distance. This is done by first determining n' on the load-line, using the auxiliary polygon $A'a'B'$, to a pole O' (arbitrary), and drawing $O'n'$ ∥ to $A'B'$. Taking O'' on a horizontal through n', making $O''n' = H''$, we complete the force diagram, and equil. pol. A a B. Then, z being the vertical intercept between m and the equil. polygon, we have

$$\text{Moment at } m = M = H''z \quad (\text{or} = H'z' \text{ also})$$

and shear at m, or J, $= 2 \ldots n'$, i.e. = projection of the proper ray R_2, or $O'' \ldots 2$, upon the vertical through m. Similarly we obtain M and J at any other section for the given load. (See §§ 329, 337, and 357)
The moment of inertia need not be constant in this case.

389. STRAIGHT HORIZONTAL GIRDER of FIXED ENDS.

No horizontal constraint hence no thrust. I constant. Ends at same level and fixed with horiz. tangents. We may consider the whole beam free (cutting close to the walls) putting in the unknown upward shears J_0 and J_n, and the two stress couples of unknown moments M_0 and M_n at these end sections; Also, as in §388, an arbitrary H'' horizontal and in line of beam at each extremity. Now the couple at O and the force H'' are equivalent to a single horiz. H'' at an



unknown vertical distance c below O; similarly at the right hand end. The spec. pol. FG is to be determined for this new system. Since the end tangents are fixed $\sum M \Delta s = 0 \therefore \sum_0^B z \Delta s = 0$ and since O's displacement relatively to B's tangent is zero we have $\sum M x \Delta s = 0 \therefore \sum H'' z x \Delta s = 0 \quad \therefore \sum xz \Delta s = 0$ hence for EQUAL Δs's, $\Sigma(z) = 0$ and $\Sigma(xz) = 0$ (§377). Now for any pole O''' draw an equil. pol. F''' G''' and in it locate v''' m''' so as to make $\Sigma(z''') = 0$ and $\Sigma(xz''') = 0$. Draw verticals thro' the intersections E''' and L''', to determine E and L on the beam; these are points in pol. FG. Draw O'''n'' ∥ to v''' m''' to fix n''. Take a pole O'' on the horiz. thro' n'', making O'' n'' = H'' (arbitrary), draw the force diagram O'' 1 2 3 4 and a corresponding equil. pol. beginning at E. It should cut L. From this pol. FG may be obtained the moms.; and shears from the force diagram